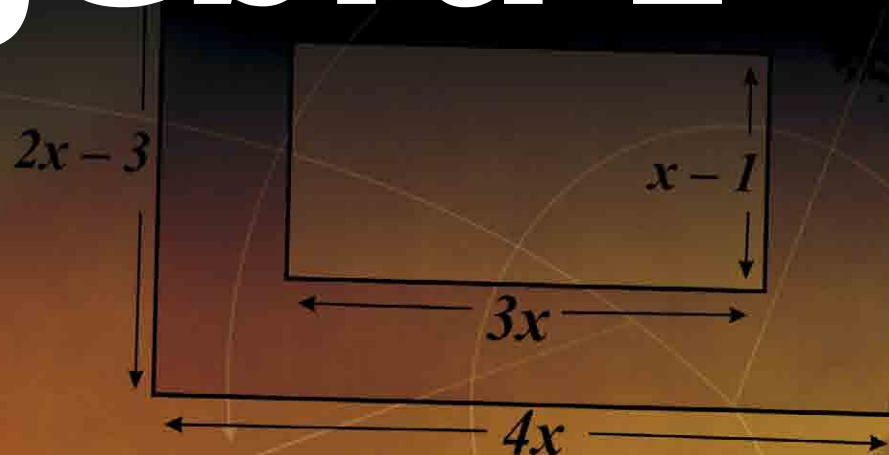


California Algebra I



$$A = 4x(2x - 3) - 3x(x - 1)$$

$$A = 8x^2 - 12x - 3x^2 + 3x$$

$$A = 5x^2 - 9x$$

$$A = x(5x - 9)$$

Student Textbook

California Standards-Driven Program

California **Algebra I**

Student Textbook

California Standards-Driven Program



Contents

This Textbook provides comprehensive coverage of all the California Algebra I Standards. The Textbook is divided into eight Chapters. Each of the Chapters is broken down into small, manageable Topics and each Topic covers a specific Standard or part of a Standard.

California Standard

Chapter 1 — Working with Real Numbers

Section 1.1 — Sets and Expressions

1.0	Topic 1.1.1	The Basics of Sets	2
	Topic 1.1.2	Subsets of the Real Numbers	5
	Topic 1.1.3	Unions and Intersections	7
	Topic 1.1.4	Algebraic and Numeric Expressions	9
	Topic 1.1.5	Coefficients and Evaluating	12

Section 1.2 — The Real Number System

1.0	Topic 1.2.1	The Number System	14
2.0	Topic 1.2.2	Identities and Inverses	16
	Topic 1.2.3	The Number Line and Absolute Values	18
	Topic 1.2.4	Addition and Multiplication	21
	Topic 1.2.5	Subtraction and Division	24
	Topic 1.2.6	Order of Operations	27
	Topic 1.2.7	Properties of Real Numbers	30
	Topic 1.2.8	More on Multiplication	32
	Topic 1.2.9	The Real Number Axioms	34

Section 1.3 — Exponents, Roots, and Fractions

2.0	Topic 1.3.1	Exponent Laws	37
	Topic 1.3.2	Square Roots	40
	Topic 1.3.3	Properties of Roots	43
	Topic 1.3.4	Equivalent Fractions	46
	Topic 1.3.5	Multiplying and Dividing Fractions	49
	Topic 1.3.6	Adding and Subtracting Fractions	52

Section 1.4 — Mathematical Logic

24.0	Topic 1.4.1	Mathematical Proofs	55
25.0	Topic 1.4.2	Inductive and Deductive Reasoning	58
	Topic 1.4.3	Algebraic Statements	61

Chapter 1 — Investigation

	Counting Collections	65
--	----------------------------	----

Chapter 2 — Single Variable Linear Equations

4.0	Section 2.1 — Algebra Basics	
	Topic 2.1.1 Simplifying Algebraic Expressions	67
	Topic 2.1.2 Getting Rid of Grouping Symbols	70
	Topic 2.1.3 More Simplifying and Checking Answers	73
4.0	Section 2.2 — Manipulating Equations	
	Topic 2.2.1 Properties of Equality	76
4.0	Section 2.3 — More Equations	
	Topic 2.3.1 Removing Fractions	81
	Topic 2.3.2 Fractional Coefficients in Algebraic Expressions	84
	Topic 2.3.3 Eliminating Decimal Coefficients	86
4.0 5.0	Section 2.4 — Using Equations	
	Topic 2.4.1 Applications of Linear Equations	89
	Topic 2.4.2 Coin Tasks	92
4.0 5.0 15.0	Section 2.5 — Consecutive Integer Tasks, Time and Rate Tasks	
	Topic 2.5.1 Consecutive Integer Tasks	95
	Topic 2.5.2 Age-Related Tasks	99
	Topic 2.5.3 Rate, Time, and Distance Tasks	103
15.0	Section 2.6 — Investment and Mixture Tasks	
	Topic 2.6.1 Investment Tasks	108
	Topic 2.6.2 Mixture Tasks	113
	Topic 2.6.3 More Mixture Tasks	116
15.0	Section 2.7 — Work-Related Tasks	
	Topic 2.7.1 Work Rate	121
	Topic 2.7.2 Combined Work Rate	124
3.0	Section 2.8 — Absolute Value Equations	
	Topic 2.8.1 Absolute Value Equations	128
	Topic 2.8.2 More on Absolute Value	130
	Chapter 2 — Investigation	
	Wildlife Park Trains	135

Chapter 3 — Single Variable Linear Inequalities

4.0 5.0	Section 3.1 — Inequalities	
	Topic 3.1.1 Inequalities	137
	Topic 3.1.2 Addition and Subtraction Properties of Inequalities	139
	Topic 3.1.3 Multiplication and Division Properties of Inequalities	142

Contents

California Standard

Chapter 3 — Continued

Section 3.2 — Applications of Inequalities

4.0	Topic 3.2.1	Multistep Inequalities	147
5.0	Topic 3.2.2	Applications of Inequalities	150

Section 3.3 — Compound Inequalities

4.0	Topic 3.3.1	Compound Inequalities	155
-----	-------------	-----------------------------	-----

Section 3.4 — Absolute Value Inequalities

3.0	Topic 3.4.1	Absolute Value Inequalities	158
-----	-------------	-----------------------------------	-----

Chapter 3 — Investigation

	Mailing Packages	162
--	------------------------	-----

Chapter 4 — Linear Equations & Their Graphs

Section 4.1 — The Coordinate Plane

6.0	Topic 4.1.1	The Coordinate Plane	164
	Topic 4.1.2	Quadrants of the Plane	167
	Topic 4.1.3	Lines	170
	Topic 4.1.4	Horizontal and Vertical Lines	173

Section 4.2 — Lines

6.0	Topic 4.2.1	Points on a Line	175
7.0	Topic 4.2.2	Graphing $Ax + By = C$	177
	Topic 4.2.3	Using Line Equations	179
	Topic 4.2.4	The x - and y -intercepts	182
	Topic 4.2.5	Graphing Lines	186

Section 4.3 — Slope

7.0	Topic 4.3.1	Slope of a Line	189
	Topic 4.3.2	Point-Slope Formula	194

Section 4.4 — More Lines

8.0	Topic 4.4.1	Parallel Lines	197
	Topic 4.4.2	Perpendicular Lines	201
	Topic 4.4.3	The Slope-Intercept Form of a Line	205
	Topic 4.4.4	More About Slopes	209

Section 4.5 — Inequalities

6.0	Topic 4.5.1	Regions Defined by Inequalities	212
9.0	Topic 4.5.2	Borders of Regions	217
	Topic 4.5.3	Regions Defined by Two Inequalities	222

Chapter 4 — Investigation

	Tree Growth	227
--	-------------------	-----

Chapter 5 — Systems of Equations

Section 5.1 — Systems of Equations

9.0	Topic 5.1.1	The Graphing Method	229
	Topic 5.1.2	The Substitution Method	232
	Topic 5.1.3	Comparing the Graphing and Substitution Methods	234
	Topic 5.1.4	Inconsistent Systems	237
	Topic 5.1.5	Dependent Systems	239
	Topic 5.1.6	Systems of Equations — Further Examples	241

Section 5.2 — The Elimination Method

9.0	Topic 5.2.1	Systems of Equations — Elimination	244
-----	-------------	--	-----

Section 5.3 — Applications of Systems of Equations

9.0	Topic 5.3.1	Systems of Linear Equations — Applications	249
15.0	Topic 5.3.2	Systems of Linear Equations — Integer Problems	253
	Topic 5.3.3	Systems of Linear Equations — Percent Mix Problems	255
	Topic 5.3.4	Systems of Linear Equations — Rate Problems	258

Chapter 5 — Investigation

	Breaking Even in an Egg Business	261
--	--	-----

Chapter 6 — Manipulating Polynomials

Section 6.1 — Adding and Subtracting Polynomials

2.0	Topic 6.1.1	Polynomials	263
10.0	Topic 6.1.2	Adding Polynomials	266
	Topic 6.1.3	Subtracting Polynomials	269
	Topic 6.1.4	Adding and Subtracting Polynomials	272

Section 6.2 — Multiplying Polynomials

2.0	Topic 6.2.1	Rules of Exponents	274
10.0	Topic 6.2.2	Polynomial Multiplication	276
	Topic 6.2.3	Polynomial Multiplication - Area and Volume	280

Section 6.3 — Dividing Polynomials

2.0	Topic 6.3.1	Division by Monomials	283
10.0	Topic 6.3.2	Polynomials and Negative Powers	285
	Topic 6.3.3	Division by Polynomials — Factoring	288
	Topic 6.3.4	Division by Polynomials — Long Division	291
	Topic 6.3.5	Division by Polynomials — Applications	294

Section 6.4 — Special Products of Binomials

2.0	Topic 6.4.1	Special Products of Two Binomials	297
10.0			

Contents

California Standard

Chapter 6 — Continued

Section 6.5 — Factors

11.0	Topic 6.5.1	Factors of Monomials	302
	Topic 6.5.2	Simple Factors of Polynomials	305

Section 6.6 — Factoring Quadratics

11.0	Topic 6.6.1	Factoring Quadratics	310
	Topic 6.6.2	Factoring Quadratics — $ax^2 + bx + c$	313

Section 6.7 — More on Factoring Polynomials

11.0	Topic 6.7.1	Factoring Quadratic Expressions in Two Variables	318
	Topic 6.7.2	Factoring Third-Degree Polynomials	322

Section 6.8 — More on Quadratics

11.0	Topic 6.8.1	The Difference of Two Squares	325
	Topic 6.8.2	Perfect Square Trinomials	327
	Topic 6.8.3	Factoring by Grouping	329

Chapter 6 — Investigation

	Pascal's Triangle	331
--	-------------------------	-----

Chapter 7 — Quadratic Equations & Their Applications

Section 7.1 — Solving Quadratic Equations

11.0 14.0	Topic 7.1.1	Solving Quadratic Equations by Factoring	333
	Topic 7.1.2	Quadratic Equations — Taking Square Roots	338

Section 7.2 — Completing the Square

14.0	Topic 7.2.1	Completing the Square	342
	Topic 7.2.2	More on Completing the Square	347
	Topic 7.2.3	Solving Equations by Completing the Square	351

Section 7.3 — The Quadratic Formula and Applications

19.0 20.0	Topic 7.3.1	The Quadratic Formula	355
	Topic 7.3.2	Applications of Quadratics	359

Section 7.4 — Quadratic Graphs

21.0 22.0	Topic 7.4.1	Graphs of Quadratic Functions	362
	Topic 7.4.2	Drawing Graphs of Quadratic Functions	365
	Topic 7.4.3	Quadratic Graphs and Completing the Square	368

Section 7.5 — The Discriminant

22.0	Topic 7.5.1	Quadratic Equations and the Discriminant	372
------	-------------	--	-----

Section 7.6 — Motion Tasks and Other Applications

23.0	Topic 7.6.1	Motion Tasks	379
	Topic 7.6.2	Economic Tasks	383

Chapter 7 — Continued

Chapter 7 — Investigation

The Handshake Problem	386
-----------------------------	-----

Chapter 8 — Rational Expressions & Functions

Section 8.1 — Rational Expressions

12.0	Topic 8.1.1	Fractions and Rational Expressions	388
	Topic 8.1.2	Equivalent Fractions	390

Section 8.2 — Multiplying and Dividing Rational Expressions

13.0	Topic 8.2.1	Multiplying Rational Expressions	394
	Topic 8.2.2	Dividing Rational Expressions	397

Section 8.3 — Adding and Subtracting Rational Expressions

13.0	Topic 8.3.1	Fractions with Identical Denominators	403
	Topic 8.3.2	Fractions with Different Denominators	406

Section 8.4 — Solving Equations with Fractions

13.0	Topic 8.4.1	Solving Fractional Equations	412
	Topic 8.4.2	Applications of Fractional Equations	417

Section 8.5 — Relations and Functions

16.0	Topic 8.5.1	Relations	420
17.0	Topic 8.5.2	Functions	424
18.0	Topic 8.5.3	Function Notation	429
	Topic 8.5.4	More on Functions	431

Chapter 8 — Investigation

Transforming Functions	435
------------------------------	-----

Appendixes

Glossary	438
Formula Sheet	440
Index	442

California Algebra I Standards

The following table lists all the California Mathematics Content Standards for Algebra I with cross references to where each Standard is covered in this Textbook. Each Topic begins by quoting the relevant Standard in full, together with a clear and understandable objective. This will enable you to measure your progression against the California Algebra I Standards as you work your way through the Program.

California Standard	Standard Description	
1.0	Students identify and use the arithmetic properties of subsets of integers and rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable:	Chapter 1
1.1	Students use properties of numbers to demonstrate whether assertions are true or false.	Chapter 1
2.0	★ Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.	Chapters 1, 6
3.0	Students solve equations and inequalities involving absolute values.	Chapters 2, 3
4.0	★ Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.	Chapters 2, 3
5.0	★ Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.	Chapters 2, 3
6.0	★ Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequalities (e.g., they sketch the region defined by $2x + 6y < 4$).	Chapter 4
7.0	★ Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.	Chapter 4
8.0	Students understand the concepts of parallel lines and perpendicular lines and how their slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.	Chapter 4
9.0	★ Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.	Chapters 4, 5
10.0	★ Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.	Chapter 6
11.0	Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.	Chapters 6, 7
12.0	★ Students simplify fractions with polynomials in the numerator and denominator by factoring both and reducing them to the lowest terms.	Chapter 8
13.0	★ Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.	Chapter 8

14.0	★ Students solve a quadratic equation by factoring or completing the square.	Chapter 7
15.0	★ Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.	Chapters 2, 5
16.0	Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.	Chapter 8
17.0	Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.	Chapter 8
18.0	Students determine whether a relation defined by a graph, a set of ordered pairs, or a symbolic expression is a function and justify the conclusion.	Chapter 8
19.0	★ Students know the quadratic formula and are familiar with its proof by completing the square.	Chapter 7
20.0	★ Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.	Chapter 7
21.0	★ Students graph quadratic functions and know that their roots are the x -intercepts.	Chapter 7
22.0	Students use the quadratic formula or factoring techniques or both to determine whether the graph of a quadratic function will intersect the x -axis in zero, one, or two points.	Chapter 7
23.0	★ Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.	Chapter 7
24.0	Students use and know simple aspects of a logical argument:	} Illustrated in Chapter 1 and throughout Program
24.1	Students explain the difference between inductive and deductive reasoning and identify and provide examples of each.	
24.2	Students identify the hypothesis and conclusion in logical deduction.	
24.3	Students use counterexamples to show that an assertion is false and recognize that a single counterexample is sufficient to refute an assertion.	
25.0	Students use properties of the number system to judge the validity of results, to justify each step of a procedure, and to prove or disprove statements:	
25.1	Students use properties of numbers to construct simple, valid arguments (direct and indirect) for, or formulate counterexamples to, claimed assertions.	
25.2	Students judge the validity of an argument according to whether the properties of the real number system and the order of operations have been applied correctly at each step.	
25.3	Given a specific algebraic statement involving linear, quadratic, or absolute value expressions or equations or inequalities, students determine whether the statement is true sometimes, always, or never.	

★ Shows key standards



Published by CGP Education

Editors: Tim Major and Andy Park

California Special Needs Advisor: Teresa J. Miller

Teresa is a Mathematics Teacher in Capistrano Unified School District. She has a master's degree in Education with an emphasis on Special Education. She holds two teaching credentials: Foundation Level Mathematics and Resource Specialist.

General Advisor: James Schierer

James is a High School Mathematics Teacher for the King City Joint Union High School District. He has a bachelor's degree in History. He holds two teaching credentials: Mathematics and History.

General Advisor: Daisy Lee

Daisy is a High School Mathematics Teacher in Los Angeles Unified School District. She has a bachelor's degree in Mathematics.

Additional Material: Clarence E. Davis, Ph.D

Clarence is an educational consultant for Middle Grade Mathematics and was an Assistant Professor of Mathematics Education.

Additional Material: Rosemarie Ingles Seraphin

Rosemarie is a Middle School and High School Mathematics Teacher, Fairfax County, Virginia.

Supporting Editors:

Chris Dennett	Glenn Rogers
Sarah Hilton	Emma Stevens
Kate Houghton	Ami Snelling
Sharon Keeley	Claire Thompson
Ali Palin	Julie Wakeling
Alan Rix	Tim Wakeling

Proofreading: Judith Curran Buck, Jennifer Heller, Amanda Jones, Rafiq Ladhani, Betsey McCarty and Cathy Podeszwa.

Graphic Design: Caroline Batten, Russell Holden, Jane Ross and Ash Tyson.

Mathematics Content Standards for California Public Schools reproduced by permission, California Department of Education, CDE Press, 1430 N Street, Suite 3207, Sacramento, CA 95814.

ISBN 13: 978 1 60017 014 0

website: www.cgpeducation.com

Wakeling Publishing Services: www.wakelingpublishingservices.co.uk

Printed by Elanders Hindson Ltd, UK and Johnson Printing, Boulder, CO

Clipart sources: CorelDRAW and VECTOR.

Text, design, layout, and illustrations © CGP, Inc. 2007
All rights reserved.

Chapter 1

Working with Real Numbers

Section 1.1	Sets and Expressions	2
Section 1.2	The Real Number System	14
Section 1.3	Exponents, Roots, and Fractions	37
Section 1.4	Mathematical Logic	55
Investigation	Counting Collections	65

Section 1.1 The Basics of Sets

California Standards:

1.0: Students identify and use the arithmetic properties of **subsets of integers** and rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable.

What it means for you:

You'll learn about sets and subsets.

Key words:

- set
- element
- subset
- universal set
- empty set

Check it out:

In this course the universal set consists of all real numbers (see Topic 1.1.2), unless otherwise specified.

*Sets are a really useful way of being able to say whether numbers or variables have something in **common**. There's some new notation here, but the math isn't too hard at all.*

Sets are Collections of Elements

A **set** is a collection of objects. Each object in the set is called an **element** or a **member**. You use the symbol \in to show that something is a member of a set — you read it as “is an element of” or “is a member of.” The symbol \notin means the opposite — you read it as “is not an element of” or “is not a member of.”

Sets are usually named by **capital letters**. The elements of a set are enclosed in **braces** $\{ \}$. For example, $A = \{1, 3, 5, 7\}$ is a set containing 4 elements — the numbers 1, 3, 5, and 7.

The **universal set**, denoted by the symbol \mathbb{U} , is the set of all objects under consideration (so in a math course, the universal set often consists of all numbers).

Example 1

Given that set $A = \{1, x, 2, b\}$, determine whether each of the following statements is true or false.

- a) $x \in A$ b) $2 \in A$ c) $1 \notin A$ d) $3 \in A$

Solution

- a) x is an element of A , so the statement is **true**.
 b) 2 is an element of A , so the statement is **true**.
 c) 1 is an element of A ($1 \in A$), so the statement is **false**.
 d) 3 is not an element of A , so the statement is **false**.

✓ Guided Practice

Given that set $A = \{x, 2, 4, y\}$, determine whether each of the following statements is true or false.

1. $y \in A$
2. $2 \in A$
3. $6 \notin A$
4. $4 \notin A$

The Empty (or Null) Set has No Elements

An **empty set** (or **null set**) is a set without any elements or members. It's denoted by \emptyset or $\{ \}$.

Subsets are Contained Within Other Sets

A **subset** is a set whose elements are also contained in another set. The symbol \subset means “is a subset of.”

Any set is a subset of itself — and the empty set is a subset of **any** set.

Example 2

Let $A = \{\text{all odd numbers}\}$ and $B = \{1, 3, 5, 7\}$. Is B a subset of A ?

Solution

Work through the elements of B one by one:

$1 \in A$, $3 \in A$, $5 \in A$, and $7 \in A$, because they are all odd numbers.

All elements of B are also elements of A — so **$B \subset A$** .

Example 3

Let $A = \{0, 1, 2\}$. Determine all the subsets of set A .

Solution

List the empty set first because it's a subset of any set. Then write all the subsets with 1 element, then all the subsets with 2 elements, and so on until finally you finish with the whole set.

So the subsets of A are:

$\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\},$ and $\{0, 1, 2\}$

✓ Guided Practice

Let $A = \{a, 2, 3, b, c\}$ and $B = \{1, 2, c, d\}$.

Use sets A and B to answer questions 5 and 6:

5. Explain whether $B \subset A$.
6. List all the subsets of set B .

Let $C = \{\text{all prime numbers less than 13 but greater than 7}\}$.

7. List set C and all its subsets.

For exercises 8 and 9, let $M = \{\text{all real numbers } b \text{ such that } b = 3x - 1\}$.

8. List the members of set M if $x \in \{1, 2, 3, 5\}$.
9. List the subsets of set M if $x \in \{1, 2, 3, 5\}$.

Equality of Sets

Check it out:

Another way of saying that set A and B are equal is that

- A is a subset of B...
- ...and B is a subset of A.

Two sets are equal if they have all of the same elements in them.

So if $A = \{1, 3, 5\}$ and $B = \{3, 5, 1\}$, then A and B are equal.

Or more mathematically, two sets A and B are said to be **equal** if every element in set A is in set B, and every element in set B is in set A.

Note also that all **empty sets** are equal — because they're exactly the same. That's why you say **the** empty set, not **an** empty set.

✓ Independent Practice

1. Determine the set H whose elements are all the multiples of both 2 and 3 that are less than 30 but greater than 12.

2. Let $U = \{\text{all letters of the alphabet}\}$. Determine set V, whose elements are the vowels.

3. Write down $G = \{\text{all prime numbers greater than 11 but less than 13}\}$.

For exercises 4 and 5, let $F = \{\text{red, yellow, blue, purple}\}$, and let $G \subset F$ and $H \subset F$.

4. Determine set G, whose elements are 3-letter words in F.

5. Determine set H, whose elements are 5-letter words in F.

For exercises 6 and 7, let $A = \{\text{all even numbers}\}$ and $B = \{2, 4, 6, 8\}$.

6. Explain whether $B \subset A$.

7. List all the subsets of B.

For exercises 8 and 9, let $M = \{3, a, 9, b, 15\}$ and $N = \{3, 6, 9\}$.

8. Explain whether $N \subset M$.

9. List all the subsets of N.

10. Let $C = \{\text{all even numbers less than 10 but greater than 2}\}$.

List set C and all its subsets.

11. Let $J = \{\text{all real numbers } y \text{ such that } y = 2x + 3\}$.

List the members of set J if $x \in \{0, 1, 2, 3\}$.

12. Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{\text{all odd numbers less than 10 but greater than 0}\}$. Explain whether $A = B$.

Round Up

*Sets sound a little odd but they're just a way of **grouping together** types of numbers or variables. You've seen a lot of the stuff in this Topic in earlier grades, but in Algebra I you've got to treat everything formally and give the **proper names** for things like the empty set.*

Subsets of the Real Numbers

California Standards:

1.0: Students identify and use the arithmetic properties of **subsets of integers and rational, irrational, and real numbers**, including closure properties for the four basic arithmetic operations where applicable.

What it means for you:

You'll learn about the set of real numbers and its subsets.

Key words:

- real numbers
- natural numbers
- whole numbers
- integers
- rational numbers
- irrational numbers
- set
- element
- subset

Check it out:

The natural numbers (\mathbb{N}) are sometimes called the counting numbers, because they're the numbers you use in everyday life to count things.

Check it out:

The symbol $\not\subset$ means "is not a subset of."

*This Topic is about one set of numbers that is really important. You'll be referring to the **real numbers** throughout Algebra I.*

The Set of Real Numbers Has Subsets

In Algebra I, the sets used are usually subsets of the **real numbers**.

\mathbb{R} — **Real Numbers** The set of real numbers consists of all positive numbers, zero, and all negative numbers.

The following are subsets of \mathbb{R} :

\mathbb{N} — **Natural numbers** $\mathbb{N} = \{1, 2, 3, \dots\}$

\mathbb{W} — **Whole numbers** $\mathbb{W} = \{0, 1, 2, 3, \dots\}$

\mathbb{Z} — **Integers** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Example 1

Explain why $\mathbb{N} \subset \mathbb{W}$.

Solution

The set \mathbb{N} of natural numbers is a subset of the set \mathbb{W} of whole numbers because every member of set \mathbb{N} is a member of set \mathbb{W} .

Example 2

Explain why $\mathbb{Z} \not\subset \mathbb{N}$.

Solution

To show this, you need to find an element in \mathbb{Z} that is not in \mathbb{N} . (You can use any negative integer or zero.)

Since $-1 \in \mathbb{Z}$, but $-1 \notin \mathbb{N}$, the set \mathbb{Z} is not a subset of \mathbb{N} ($\mathbb{Z} \not\subset \mathbb{N}$).

✓ Guided Practice

1. Determine the subset of \mathbb{N} (natural numbers) whose elements are multiples of 2 that are greater than 10 but less than 20.
2. Determine the subset of \mathbb{Z} (integers) whose elements are greater than -1 but less than 5.
3. Give an example of an element in \mathbb{R} (real numbers) that is not in \mathbb{Z} (integers).
4. Explain why \mathbb{R} (real numbers) $\not\subset \mathbb{Z}$ (integers).

Rationals and Irrationals — Two More Subsets of \mathbb{R}

Check it out:

Notice that since $q \in \mathbb{N}$,
 q cannot equal zero.

A **rational number** is a number that can be expressed in the form $\frac{p}{q}$, where p is an integer and q is a natural number.

Rational Numbers:

$$\mathbb{Q} = \left\{ \text{all numbers that can be expressed as fractions } \frac{p}{q}, \text{ where } p \in \mathbb{Z}, q \in \mathbb{N} \right\}$$

For example, 3.5 is a rational number — it can be expressed as $\frac{7}{2}$.

An **irrational number** is a number that cannot be expressed in the form $\frac{p}{q}$, where p is an integer and q is a natural number.

Irrational Numbers:

$$\mathbb{I} = \left\{ \text{all numbers that cannot be expressed as fractions } \frac{p}{q}, \text{ where } p \in \mathbb{Z}, q \in \mathbb{N} \right\}$$

For example, $\sqrt{2} = 1.4142\dots$ is an irrational number — it can't be expressed in the form $\frac{p}{q}$.

Both \mathbb{Q} and \mathbb{I} are subsets of the real numbers, \mathbb{R} .

✓ Guided Practice

5. Explain why $\mathbb{N} \subset \mathbb{Q}$.
6. Explain why $\mathbb{Z} \not\subset \mathbb{I}$.
7. Determine the subset, C , of \mathbb{Q} whose elements are also members of \mathbb{I} .

✓ Independent Practice

Remember that $\mathbb{R} = \{\text{real numbers}\}$, $\mathbb{N} = \{\text{natural numbers}\}$, $\mathbb{W} = \{\text{whole numbers}\}$, $\mathbb{Z} = \{\text{integers}\}$, $\mathbb{Q} = \{\text{rational numbers}\}$, and $\mathbb{I} = \{\text{irrational numbers}\}$.

Classify each of the following statements as true or false.

1. $\mathbb{N} \subset \mathbb{W}$
2. $\emptyset \not\subset \mathbb{N}$
3. $\mathbb{R} \subset \mathbb{Q}$
4. $\mathbb{Z} \not\subset \mathbb{Q}$
5. $\left\{ \frac{2}{3}, 0, 4, 7.25, 8.\bar{6} \right\} \subset \mathbb{Q}$

Round Up

"Real numbers" is just a formal name for all positive and negative numbers and zero — there's nothing more to it than a new name. You need to know all the common labels for the subsets.

Unions and Intersections

California Standards:

1.0: Students identify and use the arithmetic properties of **subsets of integers and rational, irrational, and real numbers**, including closure properties for the four basic arithmetic operations where applicable.

What it means for you:

You'll learn about unions and intersections of sets.

Key words:

- union
- intersection
- set
- element

Check it out:

$A \cup B$ is said "A union B."

Check it out:

$A \cap B$ is said "A intersect B" (or "A intersection B").

This Topic is all about two symbols that represent ways of grouping elements in sets — unions and intersections.

The Union of Sets is Anything in Either or Both Sets

The **union** of sets A and B, denoted $A \cup B$, is the set of all elements in **either A or B or both sets**.

Example 1

Let $A = \{4, 6, 8, 20\}$ and $B = \{6, 8, 9, 15\}$. Find $A \cup B$.

Solution

Elements are not counted twice — they are either "in or out."
So 8, which is in both A and B, only appears once in $A \cup B$.

So $A \cup B = \{4, 6, 8, 9, 15, 20\}$.

✓ Guided Practice

Let $A = \{1, 2, 4, 5\}$, $B = \{0, 3, 6\}$, $C = \{2, 4, 6, 8\}$, $D = \{1, 3, 5, 7\}$.

1. Find $A \cup B$.
2. Find $C \cup D$.
3. Find $A \cup C$.
4. Find $B \cup D$.

The Intersection of Sets is Anything in Both Sets

The **intersection** of sets A and B, denoted $A \cap B$, is the set of all elements that are in **both set A and set B**.

Example 2

Let $A = \{4, 6, 8, 20\}$ and $B = \{6, 9, 15\}$. Find $A \cap B$.

Solution

Only the number 6 is in both A and B.

So $A \cap B = \{6\}$.

Example 3

Let $A = \{\text{all even numbers}\}$ and $B = \{\text{all odd numbers}\}$. Find $A \cap B$.

Solution

There are **no** elements in both A and B, so $A \cap B = \emptyset$.

✓ Guided Practice

Let $A = \{1, 2, 4, 5\}$, $B = \{0, 3, 6\}$, $C = \{2, 4, 6, 8\}$, $D = \{1, 3, 5, 7\}$.

- Find $A \cap B$.
- Find $A \cap C$.
- Find $C \cap D$.
- Find $B \cap D$.

You Can Find Unions and Intersections for Two Sets

If you have any two sets, you can always work out the **union** and **intersection**.

Example 4

Let $A = \{2, 4, 6, 8, 10, 12\}$ and $B = \{3, 6, 9, 12, 15\}$.

Find $A \cap B$ and $A \cup B$.

Solution

$A \cap B$ is the set of all the elements that appear in both A and B .

So $A \cap B = \{6, 12\}$.

$A \cup B$ is the set of all the elements that appear in A or B , or both sets.

So $A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12, 15\}$.

✓ Independent Practice

- Explain why $A \cap B$ is a subset of A .

If $A = \{1, 2, 3, 4, 5, 11, 12\}$, $B = \{2, 4, 6, 8, 10, 11, 20\}$, and $C = \{5, 10, 15, 20, 24\}$, write down the following sets.

- $A \cap B$
- $(A \cap B) \cup C$
- $(A \cap B) \cap C$

Let $\mathbb{U} = \{\text{all } b \in \mathbb{N} \text{ such that } b \leq 20\}$, $M = \{\text{all } b \text{ in } \mathbb{U} \text{ such that } b \text{ is a multiple of } 2\}$, $V = \{\text{all } b \text{ in } \mathbb{U} \text{ such that } b \text{ is a multiple of } 3\}$, and $H = \{\text{all } b \text{ in } \mathbb{U} \text{ such that } b \text{ is a multiple of } 4\}$.

- Determine the set $M \cap V$.
- Determine the set $(M \cap V) \cap H$.
- Determine the set $(M \cup V) \cup H$.

Let $A = \{\text{prime numbers}\}$, $B = \{\text{square numbers}\}$, $C = \{\text{even numbers}\}$, $D = \{\text{odd numbers}\}$, and $E = \{\text{natural numbers less than } 10\}$.

- Copy and complete the following expressions using sets A to E :

- $A \cap \dots = \emptyset$,
- $\dots \cap \dots = \{2\}$,
- $\dots \cap D \cap E = \{1, 9\}$
- $A \cap \dots \cap \dots = \{3, 5, 7\}$

- Write down the number of subsets for each set a–d in exercise 8.
- Use your answer to exercise 9 to write down a formula for the number of subsets of a set with n members.

Round Up

The big thing to remember here is that the **union sign** is \cup and the **intersection sign** is \cap .
Make sure you understand the definitions of each way of grouping elements.

Algebraic and Numeric Expressions

California Standards:

1.1: Students use properties of numbers to demonstrate whether assertions are true or false.

What it means for you:

You'll deal with simple expressions that contain numbers and unknowns.

Key words:

- expression
- numeric expression
- algebraic expression
- variable

*Expressions are **mathematical statements**. This Topic is all about two kinds of expressions — ones that contain only numbers, and ones that contain both numbers and unknown values.*

Numeric Expressions Just Include Numbers

A **numeric expression** contains only **numbers** — for example, 2.54×2 is a numeric expression.

Since there are no **unknown quantities**, you can always work out the **value** of a numeric expression — the value of 2.54×2 is **5.08**.

Numeric expressions with the **same value** are like **different names** for the same thing.

All these expressions have the same value:

$$\begin{array}{l} 2.54 \times 2 \\ 5 + 0.08 \\ 6 - 0.92 \end{array} \begin{array}{l} \swarrow \\ \swarrow \\ \swarrow \end{array} \begin{array}{l} \text{The value of each of these} \\ \text{numeric expressions is } \mathbf{5.08}. \end{array}$$

So if you saw (2.54×2) somewhere, you could write $(5 + 0.08)$ instead, since the two expressions have the same value — they're describing the **same thing**.

✓ Guided Practice

Calculate the value of each of these numeric expressions:

- | | |
|--------------------|-----------------------|
| 1. $18.4 + 8.23$ | 4. $1790 \div 5$ |
| 2. $37.82 - 11.19$ | 5. $37 \div 37$ |
| 3. $716 \div 2$ | 6. $19284 \div 19284$ |

For each of the following, write a numeric expression that contains the number 100 and has the same value as the expression given.

- | | |
|------------------|--------------------|
| 7. $465 - 253$ | 9. 15.6×5 |
| 8. $3850 \div 5$ | 10. $24.2 + 1.5$ |

Algebraic Expressions Contain Variables

A lot of the time in math, you can use **letters** to stand for unknown amounts. So if you were not sure how many inches long your desk was, you could just call the length x inches. When you see letters in equations, they're just numbers that you don't know the value of yet.

Letters that represent unknown numbers are called **variables**.

Example 1

The length of a desk is x inches.
How long is this in centimeters? To convert a length in inches to a length in centimeters, you multiply by 2.54.

Solution

You don't know the length of the desk, so it's been called x .

If the desk was 10 inches long, it would be $2.54 \times 10 = 25.4$ cm long.

If the desk was 50 inches long, it would be $2.54 \times 50 = 127$ cm.

In the same way, a desk that is x inches long would be

$$2.54 \times x = \mathbf{2.54x \text{ cm.}}$$

You don't have enough information to write a numerical value, so leave x in the solution.

Check it out:

When you see a number and a letter written next to each other (like $2.54x$), it means they are multiplied together.

$2.54x$ is called an **algebraic expression**, as it contains an unknown quantity. The variable x represents the **unknown quantity** — the number of inches to be converted to centimeters.

An algebraic expression always contains at least one variable (and very often it contains numbers as well).

Example 2

Write an algebraic or numeric expression for each of the following:

- The total cost of an item whose price is \$15.75 plus sales tax of \$1.30.
- The number of cents in d dollars.
- Three more than twice the length, l .

Solution

a) $\$15.75 + \$1.30 = \mathbf{\$17.05}$ (since there are no unknowns, this is a numeric expression)

b) $\mathbf{100d}$ (since the number of cents is always the number of dollars multiplied by 100)

c) $\mathbf{2l + 3}$ (since twice the length l is $2 \times l = 2l$, and three more than this is $2l + 3$)

✓ Guided Practice

Write an algebraic expression representing each of the following.

11. The sum of twice m and 7.
12. The sum of three times x and 12.
13. The difference between the total cost c , and the down payment d .
14. The number of notebooks you can buy for \$75 if each notebook costs \$ n .
15. The difference between $(3x + 7)$ and t .
16. Two more than five times d .
17. Three less than the product of x and t .
18. The age of someone six years ago who is $2y$ years old now.

✓ Independent Practice

Determine whether the following expressions are numeric or algebraic.

1. 3.14×2
2. $189x \div 2$
3. $2w + 2l$
4. $656.4 - 33$

Write an algebraic or numeric expression for each of the following.

5. The difference between x and 3
6. The cost of 5 CDs at \$ x
7. The difference between the original price of a pair of jeans, p , and the sale price, s
8. The product of 2 and 4, plus 3
9. Three times the difference between t and 2
10. One-half of 200 times a number
11. The number of feet in x yards (3 feet = 1 yard)
12. The perimeter of a rectangle with length, l , and width, w .

Round Up

Don't worry if you come across an expression that contains **variables**. Variables are just letters or symbols that represent unknown numbers, and expressions containing variables follow all the **same rules** as numeric expressions.

Coefficients and Evaluating

California Standards:

1.1: Students use properties of numbers to demonstrate whether assertions are true or false.

What it means for you:

You'll find out what coefficients are, and you'll calculate the value of expressions.

Key words:

- coefficient
- expression
- numeric expression
- algebraic expression
- variable

You've just learned about variables — *coefficients* are just another element that make up a full expression. In this Topic you'll learn about evaluating expressions, which just means *calculating the value*.

A Coefficient is the Amount to Multiply a Variable

In the algebraic expression $2.54x$, the number 2.54 is the **coefficient** of x . A coefficient is a number that **multiplies** a variable.

Example 1

Find the coefficients in these expressions:

- the coefficient of x in the algebraic expression $2x$
- the coefficient of v in $v + 5$
- the coefficient of y in $7 - 6y$
- the coefficient of k in $5 - k$
- the coefficient of m in $5(3 - 2m)$

Solution

- x is multiplied by 2, so the coefficient is **2**.
- v is the same as $1v$, so the coefficient is **1**.
- The minus sign is part of the coefficient as well — so the coefficient is **-6**.
- There's a minus sign here too — the coefficient is **-1**.
- First multiply out the parentheses: $5(3 - 2m) = 15 - 10m$
Now you can see that the coefficient is actually **-10**.

✓ Guided Practice

Find the coefficient of the variable in each of these expressions:

- $4a - 15$
- $15 - 4z$
- $x - 3$
- $2 - k$
- $2(3b + 1)$
- $4(2 - m)$

Always Write Expressions in Their Simplest Form

Different numeric expressions can have the same value.

The expressions $4 + 3$ and $14 \div 2$ both have the same value of 7. You can think of these expressions as **different names** for the number 7.

When you simplify an expression, you replace it with its simplest name. So you could simplify “ $4 + 3$ ” or “ $14 \div 2$ ” by writing “7” instead.

✓ Guided Practice

Simplify each expression below.

7. $11 + 4$

9. $24 \div 6$

11. $12a + 2 - 8a - 5$

8. 10×3.14

10. $10a + 4a$

12. $28b \div 28$

Evaluating Algebraic Expressions

Don't forget:

Variables are letters that represent unknown quantities.

You've already seen **algebraic expressions** — they're expressions containing **variables**.

But if you know the value of the letter, you can **evaluate** the expression — that means you calculate its value.

For example, the algebraic expression $2.54x$ contains the variable x . When $x = 33$, the expression $2.54x$ is equal to $2.54 \times 33 = 83.82$.

That's what is meant by “evaluating an algebraic expression” — finding the value of the expression when any variables are replaced by **specific numbers**.

✓ Guided Practice

Evaluate each of these expressions when $a = 2$, $b = 7$, and $c = -4$.

13. $a + b$

15. $c \div a$

17. $b - a$

14. $b \times a$

16. $a + b + c$

18. $b - c$

✓ Independent Practice

- Find the coefficient of x in the algebraic expression $8 - 5x$.
- Evaluate $8 - 5x$ when $x = 3$.
- Find the coefficient of k in $4.18k + 2$.
- Evaluate $10x + 2$ when $x = 0$.
- Evaluate $2x + 3y$ when $x = 10$ and $y = -1$.
- Evaluate $2x + 3y - z$ when $x = 5$, $y = 2$, and $z = 4$.
- The formula $C = \frac{5}{9}(F - 32)$ is used to convert temperatures from degrees Fahrenheit ($^{\circ}\text{F}$) to degrees Celsius ($^{\circ}\text{C}$). What is 131°F in $^{\circ}\text{C}$?

Round Up

A coefficient is just the **number in front of a variable**. It doesn't matter whether an expression includes just numbers, or numbers and variables — you always follow the same rules when you simplify.

The Number System

California Standards:

1.0: Students identify and use the arithmetic properties of subsets of integers and rational, irrational, and **real numbers, including closure properties for the four basic arithmetic operations where applicable.**

What it means for you:

You'll learn some of the rules that the real numbers always follow.

Key words:

- binary operation
- equality
- real numbers
- substitution

You've already met the real numbers in Section 1.1.

Now it's time to look at the *properties* of the real numbers in detail.

The Real Number System is Based on Simple Rules

The rules of the real number system are based on the existence of a **set of numbers**, plus two **binary operations**. A binary operation allows you to combine **two numbers** in some way to produce a third number.

In Algebra I, the set of numbers used is the real numbers, \mathbb{R} , and the two binary operations are **addition** and **multiplication**.

The Properties of Equality Always Hold

The following statements about equality hold true for any real numbers a , b , and c :

For any number $a \in \mathbb{R}$: $a = a$

This is the reflexive property of equality.

For any numbers $a, b \in \mathbb{R}$: **if $a = b$ then $b = a$**

This is the symmetric property of equality.

For any numbers $a, b, c \in \mathbb{R}$: **if $a = b$ and $b = c$, then $a = c$**

This is the transitive property of equality.

✓ Guided Practice

Name the property of equality being used in each statement.

1. If $a = 3$ then $3 = a$.
2. If $a = 3$ and $3 = b$, then $a = b$.
3. $3 = 3$
4. If $3x = 2$ and $2 = 2y$, then $3x = 2y$.
5. If $3x = 2y$ then $2y = 3x$.

Check it out:

For addition, the inputs are called **addends**, and for multiplication they are called **factors**.

Addition and Multiplication are Binary Operations

The operations of addition and multiplication are called **binary operations**. In order to carry either of them out you need to have **two "inputs."** What this basically means is that you can combine **two** numbers to produce a third number.

A set of numbers is said to be **closed** under a given binary operation if, when you perform that operation on any two members of the set, the **result is also a member of the set**.

Check it out:

Compare this to the set of irrational numbers, \mathbb{I} , which is not closed under multiplication, since $\sqrt{2} \times \sqrt{2} = 2$, where $\sqrt{2} \in \mathbb{I}$, but $2 \notin \mathbb{I}$.

Check it out:

Multiplication of a and b can be written as ab , $a \cdot b$, or $a \times b$. They all mean the same thing.

The set of real numbers, \mathbb{R} , is **closed** under both **addition** and **multiplication** — adding and multiplying real numbers always produces other real numbers.

If $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $(a + b) \in \mathbb{R}$ and $(a \times b) \in \mathbb{R}$.

Example 1

Use the fact that $10 \in \mathbb{R}$ and $6 \in \mathbb{R}$ to explain why $16 \in \mathbb{R}$ and $60 \in \mathbb{R}$.

Solution

$10 \in \mathbb{R}$ and $6 \in \mathbb{R}$, so you can add the two numbers to produce another number that is a member of \mathbb{R} . So $10 + 6 = 16 \in \mathbb{R}$.

You can also multiply the two numbers to produce another number that is a member of \mathbb{R} . So $10 \times 6 = 60 \in \mathbb{R}$.

The Substitution Principle (or Substitution Property)

For any real numbers a and b , the number a may be substituted for the number b if $a = b$.

This principle means that if two expressions have the same value, then one expression can be **substituted for** (written instead of) the other.

For example, the expression **23** can be replaced by the expression **(10 + 13)** or by **(17 + 6)**.

Similarly, the expression **2.3×4.5** can be replaced with **10.35**.

✓ Independent Practice

Determine whether each of the following statements is true or false. If false, rewrite the statement so that it is true.

1. If $k = l$ then $l = k$. This is the symmetric property.
2. If $k = l$ and $l = 3$, then $k = 3$. This is the reflexive property.

In Exercises 3–6, demonstrate the closed nature of the set under both addition and multiplication using the numbers given.

3. \mathbb{R} using 3 and 5
4. \mathbb{R} using -10 and 11
5. \mathbb{N} using 6 and 8
6. \mathbb{Q} using 1.5 and 0.3

Round Up

These rules are a bit abstract at the moment, but don't worry. You'll see how useful the real number properties are in the rest of the Section, and you'll use them all the way through Algebra I.

Identities and Inverses

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

What it means for you:

You'll find out about identities, which don't change your original value, and inverses, which take the opposite.

Key words:

- identity
- inverse

Check it out:

The additive inverse is sometimes called a negative, or an opposite.

The numbers 0 and 1 are special numbers, since they are the *identities* of addition and multiplication.

0 and 1 are Identities of Addition and Multiplication

In the real number system, there's an **additive identity**. This means that there's a number (called **zero**) which you can add to **any** number without changing the value of that number.

$$\text{For any number } a \in \mathbb{R}: \\ \mathbf{0 + a = a = a + 0}$$

There's also a **multiplicative identity**. This is a number (called **one**) which any number can be multiplied by without its value being changed.

$$\text{For any number } a \in \mathbb{R}: \\ \mathbf{1 \times a = a = a \times 1}$$

The Additive Inverse is What You Add to Make 0

Every real number has an **additive inverse** — “additive inverse” is just a more mathematical term for the “negative” of a number. When you **add** a number to its **additive inverse**, the result is **0** (the **additive identity**).

Or, more formally:

For every real number m , there is an **additive inverse** written $-m$. When you **add** a number to its additive inverse, you get **0** — which is the **additive identity**.

$$\text{So, for any number } m \in \mathbb{R}: \mathbf{-m + m = 0 = m + (-m)}$$

The inverse of a negative number is a positive number — that is, $-(-m) = m$, for any m . And if you **take the opposite** of a number, you change it into its additive inverse.

✓ Guided Practice

Find the additive inverse of the following:

- | | |
|---|--------------|
| 1. 6 | 5. $-k$ |
| 2. 81 | 6. $a + 5$ |
| 3. -4 | 7. $-5a - 2$ |
| 4. y | 8. $2ab$ |
| 9. State the value of x if $51 + x = 0$. | |
| 10. State the value of x if $x - 5 = 0$. | |

Multiply by the Multiplicative Inverse to Make 1

Check it out:

The multiplicative inverse is sometimes called the reciprocal.

Check it out:

See Topic 1.2.5 for more information about not being able to divide by zero.

Multiplication has a property that's similar to the property of addition:

Every real number also has a **multiplicative inverse** (or “reciprocal”). When you multiply a number by its multiplicative inverse, the result is **1** — the **multiplicative identity**.

However, there's an important exception: **zero has no reciprocal** — its reciprocal cannot be defined. That means that you **can't divide by zero**.

More formally this becomes:

For every nonzero real number m , there is a **multiplicative inverse** written m^{-1} . When you **multiply** a number by its multiplicative inverse, you get **1** — which is the **multiplicative identity**.

So, for any number $m \in \mathbb{R}$, $m \neq 0$: $m^{-1} \times m = 1 = m \times m^{-1}$

If m is a nonzero real number then its reciprocal is m^{-1} , which is given by

$m^{-1} = \frac{1}{m}$. And the reciprocal of m^{-1} is m — that is, $(m^{-1})^{-1} = m$,

which means that $\frac{1}{m^{-1}} = m$, or $(\frac{1}{1/m}) = m$.

Example 1

Find the reciprocals of: a) 3 b) 12 c) x d) 3^{-1}

Solution

a) $3^{-1} = \frac{1}{3}$ b) $12^{-1} = \frac{1}{12}$ c) $x^{-1} = \frac{1}{x}$ d) $(3^{-1})^{-1} = 3$

Guided Practice

Find the multiplicative inverse:

11. 2

12. -31

13. $\frac{1}{8}$

14. $-\frac{1}{16}$

15. $\frac{a}{b}$

16. Find the value of x if $3x = 1$.

Independent Practice

State the additive and multiplicative inverses of each expression below:

1. 6

2. -11

3. $\frac{1}{3}$

4. $\frac{1}{8}$

5. x

6. $(a + b)$

Round Up

Identity and inverse are just math names for quite simple things. The identity doesn't change your value, and inverses just mean taking the opposite, or finding the reciprocal.

The Number Line and Absolute Values

California Standards:

1.1: Students use properties of numbers to demonstrate whether assertions are true or false.

What it means for you:

You'll use the number line to show real numbers, and you'll describe them in terms of absolute values.

Key words:

- real numbers
- absolute value

Check it out:

A “graph” is just a type of “picture” of something mathematical.

Don't forget:

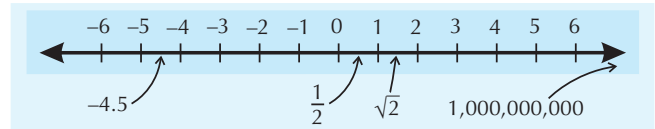
The “sign” of a number refers to whether it is positive or negative.

The *number line* is a useful way of representing numbers visually. This Topic also includes information about absolute values, which show how far from zero numbers are on the number line.

Number Lines Show Real Numbers as Points on a Line

All **real numbers**

can be found on the number line, no matter how big or small they are, and no matter whether they are rational or irrational. And for **every point** on the number line, there is a real number.



So there's a **one-to-one correspondence** between the real numbers and the points on the number line.

For each **point** on the number line, the **corresponding real number** is called the **coordinate** of the point. And for each **real number**, the **corresponding point** is called the **graph** of the number.

The numbers to the **left** of zero on the number line are all **negative** — they're **less than zero**.

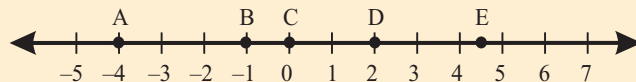
The numbers to the **right** of zero are **positive** — they're **greater than zero**. Zero is neither negative nor positive.

✓ Guided Practice

1. “Only integers can be found on the number line.”

Is this statement true or false?

2. Identify the corresponding real numbers of points A–E on the number line below.



3. Draw the graph of 6 on a number line.

4. Draw the graph of -2 on a number line.

The Number Line Can Be Useful for Calculations

If you think of the number line as a road, then you can think of coordinates as movement along the road — either to the left or to the right, depending on the coordinate’s **sign**.

For example, -5 would indicate a movement of 5 units to the left, while 4 would mean 4 units to the right.

Example 1

Find $3 - 2 + 4$.

Solution

Rewriting this as $3 + (-2) + 4$, you can interpret this as:

“Start at 3, move 2 to the left (to reach 1) and then 4 to the right (to reach 5).”

So $3 - 2 + 4 = 5$.

The Absolute Value of a Number is its Distance from 0

The opposite of a real number c (that is, $-c$) lies an equal distance from zero as c , but on the other side of zero. So the **opposite of 4** (which lies 4 units to the right of zero) **is -4** (which lies 4 units to the left of zero).

And the **opposite of -7** (which lies 7 units to the left of zero) **is 7** (which lies 7 units to the right of zero).

The distance from zero to a number is called the number’s **absolute value**. It doesn’t matter whether it is to the left or to the right of zero — so absolute value just means the “**size**” of the number, ignoring its sign.

The absolute value of c is written $|c|$.

More algebraically...

$$|c| = \begin{cases} c & \text{if } c > 0 \\ 0 & \text{if } c = 0 \\ -c & \text{if } c < 0 \end{cases}$$

The absolute value of a number can **never** be **negative**.

Example 2

Find: a) $|6|$ b) $|0|$ c) $|-23|$

Solution

a) 6 is positive, so $|6| = 6$.

b) $|0| = 0$ (by definition)

c) -23 is negative, so $|-23| = -(-23) = 23$.

✓ Guided Practice

Find the following absolute values.

5. $|6|$

6. $|15|$

7. $|-3|$

8. $|-8|$

9. $|0|$

10. $|\frac{1}{2}|$

11. $|\sqrt{2}|$

12. $|\frac{1}{3}|$

✓ Independent Practice

1. Choose the correct word from each pair to complete this sentence.

On a number line, positive numbers are found to the (left/right) of zero and negative numbers are found to the (left/right) of zero.

2. On a single number line, draw the graphs with the following coordinates: 5 -3.5 0.5

In exercises 3–11, find x .

3. $|3| = x$

4. $|-10.5| = x$

5. $|-2| = x$

6. $|3.14| = x$

7. $|-2.17| = x$

8. $|x| = 0$

9. $|465| = x$

10. $|-465| = x$

11. $|x| = 465$ (Hint: Look at exercises 9 and 10)

12. What is wrong with the equation $|x| = -1$?

Round Up

You've seen the number line plenty of times in earlier grades, but it's always useful.

*You don't always need to draw it out, but you can **imagine** a number line to work out **which direction** an operation will **move** a number.*

Addition and Multiplication

California Standards:

1.0: Students identify and use the arithmetic properties of subsets of integers and rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable.

What it means for you:

You'll see how adding or multiplying positive and negative numbers affects the sign of the result.

Key words:

- sum
- product
- addend

Don't forget:

You don't need to write the sign of a positive number — you can write +12 as 12.

All real numbers fall into one of the following three categories: *positive, zero, or negative.*

This is also true for the results of adding or multiplying real numbers (since \mathbb{R} is closed under addition and multiplication).

The Sum is What You Get When You Add Numbers

When you add two real numbers a and b (find the **sum** $a + b$), the result can be **positive**, **zero**, or **negative**, depending on a and b .

The sum of any two real numbers a and b is:

1. **Positive** if **both** addends are **positive**.

This means that $(a + b) > 0$ if $a > 0$ and $b > 0$.

This is the same as saying that the set of **positive** real numbers is **closed under addition** — so if you add two positive real numbers, you always get another positive real number.

2. **Negative** if **both** addends are **negative**.

This means that $(a + b) < 0$ if $a < 0$ and $b < 0$.

This is the same as saying that the set of **negative** real numbers is **closed under addition** — so if you add two negative real numbers, you always get another negative real number.

3. **Positive** or **Negative** or **Zero** if a and b have **opposite** signs.

In this case, the sign of $(a + b)$ is the same as the sign of the addend with the **larger absolute value**.

For example, $(-5) + 7 = 7 + (-5) = 2$ — positive, since $|7| > |-5|$.
(The positive addend has a larger absolute value, so the sum is **positive**.)

However, $(-7) + 5 = 5 + (-7) = -2$ — negative, since $|-7| > |5|$.
(The negative addend has a larger absolute value, so the sum is **negative**.)

✓ Guided Practice

State with a reason the sign of each sum, then find the sum.

1. $3 + 9$
2. $-3 + (-9)$
3. $-10 + (-5)$
4. $-3 + 9$
5. $3 + (-9)$
6. $10 + (-15)$

The Product is What You Get When You Multiply

Check it out:

The numbers being multiplied together are called factors.

In a similar way, the **sign** of the **product** of any real numbers depends on the signs of the numbers being multiplied (the factors).

The rules for the sign of the product of any **two** real numbers are as follows:

<u>Signs of the factors:</u>	<u>Sign of the product:</u>
+ +	+
- -	+
+ -	-

For example, $5 \times 2 = 10$ — positive.

The set of positive real numbers is **closed** under multiplication.

For example, $-5 \times -2 = 10$ — positive.

The set of negative real numbers is **not closed** under multiplication.

For example, $5 \times -2 = -10$ — negative.

So the product of any real numbers is:

1. **Positive** if the expression contains an **even number** of **negative factors** (or **only positive** factors).

For example, $5 \times (-2) \times 2 \times (-3) \times (-1) \times (-4) = 240$

This is positive, since the expression has four negative factors — an **even** number.

2. **Negative** if the expression contains an **odd number** of **negative factors**.

For example, $(-2) \times 3 \times (-4) \times (-2) = -48$

This is negative, since the expression has three negative factors — an **odd** number.

Guided Practice

State with a reason the sign of each product, then find the product.

- 2×6
- $7 \times (-7)$
- $(-7) \times (-7)$
- 9×10
- $(-11) \times (-3)$
- $(-1) \times 3 \times 4$

Independent Practice

State with a reason the signs of the following expressions.

- $(-5) + 3$
- $2 + (-8)$
- 5×3
- $(-10) \times (-2)$
- $(-2) \times (-1) \times (-8) \times 6 \times (-2)$

Evaluate the following numerical expressions.

- $-8 + (-2)$
- $8 + (-2)$
- $6 \times (-4)$
- $-8 + 2$
- $-8 + 10$
- $(-2) \times (-5) \times 4$
- $(-5) + 2 + 6$
- $(-3) \times (-5) \times (-2)$

State the signs of the following expressions.

- a^3 where $a < 0$.
- $-3a^2b^2$ where $a < 0$ and $b > 0$.
- $a^2b^3c^7$ where $a > 0$, $b < 0$, and $c < 0$

Round Up

*You've done addition and multiplication a lot in previous grades — but a lot of Algebra I is about making formal rules for math methods. After a while you'll carry out the **rules of changing sign** without really thinking about them — but for now you need to make sure you remember them.*

Subtraction and Division

California Standards:

1.0: Students identify and use the arithmetic properties of subsets of integers and rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable.

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

What it means for you:

You'll see how adding or multiplying positive and negative numbers affects the sign of the result.

Key words:

- difference
- quotient
- reciprocal
- commutative

Check it out:

See Topic 1.2.7 for more on what “commutative” means.

*After addition and multiplication you can probably guess what's next. This Topic gives more formal rules for **subtraction** and **division** that you've seen in earlier grades.*

Subtraction Means Adding the Additive Inverse

Subtraction is the inverse operation of addition. It is defined as the addition of the opposite of a number.

So to subtract a from b (that is, to find $b - a$), you add $-a$ to b .

$$\text{For any } a, b \in \mathbb{R}: b - a = b + (-a)$$

Finding the **difference** between two numbers means you subtract them. For example, the difference between a and b would be $a - b$ or $b - a$.

Since subtraction is a type of addition, and \mathbb{R} is closed under addition, \mathbb{R} must be **closed** under subtraction.

Subtraction is not commutative (meaning that $a - b \neq b - a$).

Example 1

Show that $a - b \neq b - a$ for any $a \neq b$.

Solution

Write both subtractions as additions:

$$a - b = a + (-b)$$

$$b - a = b + (-a) = (-a) + b$$

The two subtractions “ $a - b$ ” and “ $b - a$ ” contain different addends when they're written as additions — so $a - b \neq b - a$.

Subtracting a negative number is the same as adding a positive number (since subtracting means adding the opposite, and the opposite of a negative number is a positive number).

$$\text{For any } a, b \in \mathbb{R}: a - (-b) = a + b$$

Example 2

- a) Rewrite $11 - 7$ as an addition.
 b) Simplify $-7 - (-5)$.
 c) Find the sum $-3 + (-8)$.

Solution

- a) $11 - 7 = 11 + (-7)$
 b) As the opposite of -5 is 5 , subtracting -5 means adding 5 .
 So $-7 - (-5) = -7 + 5 = -2$.
 c) $-3 + (-8) = -(3 + 8) = -11$

✓ Guided Practice

Rewrite the following subtractions as additions:

1. $9 - 6$ 2. $10 - 4$ 3. $-4 - (-10)$

Evaluate the following:

4. $6 - (-5)$ 5. $-3 + (-10)$ 6. $-a - (-a)$

Check it out:

The reciprocal is another name for the multiplicative inverse.

Division Means Multiplying by a Reciprocal

Division is the inverse operation of multiplication. It's defined as **multiplication by a reciprocal**.

So to divide b by a (that is, to find $b \div a$), you multiply b by a^{-1} .

$$\text{For any } a, b (a \neq 0) \in \mathbb{R}: b \div a = b \times a^{-1} = b \times \frac{1}{a}$$

One number divided by another is called a **quotient**.

Division is **not commutative** (meaning that $a \div b \neq b \div a$).

Example 3

Show that $a \div b \neq b \div a$ for any $a \neq b, a, b \neq 0$.

Solution

Write both divisions as multiplications:

$$a \div b = a \times (b^{-1})$$

$$b \div a = b \times (a^{-1})$$

The two divisions " $a \div b$ " and " $b \div a$ " contain different factors when they're written as multiplications — so $a \div b \neq b \div a$.

Dividing by a number's reciprocal is the same as multiplying by the number itself.

$$\text{For any } a, b \text{ (} b \neq 0 \text{)} \in \mathbb{R}: a \div b^{-1} = \frac{a}{\left(\frac{1}{b}\right)} = a \times b$$

And lastly, **you cannot divide by 0** — since 0 has no reciprocal.

Example 4

a) Rewrite $25 \div 5$ as a multiplication.

b) Simplify $7 \div 5^{-1}$.

Solution

a) $25 \div 5 = 25 \times \frac{1}{5}$ (or 25×0.2)

b) The reciprocal of 5^{-1} is 5, so dividing by 5^{-1} means multiplying by 5.
Therefore $7 \div 5^{-1} = 7 \times 5 = 35$.

Guided Practice

Rewrite these expressions as multiplications.

7. $24 \div 6$

8. $36 \div 3$

9. $2 \div \frac{1}{8}$

Evaluate the following expressions.

10. $10 \div \frac{1}{5}$

11. $16 \div 2^{-1}$

12. $16 \div \left(\frac{1}{8}\right)^{-1}$

Independent Practice

Simplify.

1. $12 - 15$

2. $18 + (-3)$

3. $36 - (-4)$

4. $-4 - 2$

5. $20 \div (-2)$

6. $10 \div \frac{1}{2}$

7. $15 \div 3^{-1}$

8. $18 \div \left(\frac{1}{6}\right)^{-1}$

9. Use examples to demonstrate that subtraction and division are not commutative.

Round Up

Check over the part on *reciprocals* until you're sure you understand it. Although the notation is tricky, the actual ideas behind it should make sense if you read through it carefully.

Order of Operations

California Standards:

1.1: Students use properties of numbers to demonstrate whether assertions are true or false.

What it means for you:

You'll see why it's important to have a set of rules about the order in which you have to deal with operations.

Key words

- grouping symbols
- parentheses
- brackets
- braces
- exponents

It's important that all mathematicians write out expressions in the same way, so that anyone can reach the same solution by following a set of rules called the "order of operations."

Grouping Symbols Show You What to Work Out First

If you wanted to write a numerical expression representing "add 4 and 3, then multiply the answer by 2," you might be tempted to write $4 + 3 \times 2$.

But watch out — this expression contains an addition and a multiplication, and you get different answers depending on which you do first.

If you do the addition first, you get the answer $7 \times 2 = 14$.

If you do the multiplication first, you get the answer $4 + 6 = 10$.

You might know the addition has to be done first, but **somebody else** might not.

To be really clear which parts of a calculation have to be done first, you can use grouping symbols. Some common grouping symbols are: **parentheses (), brackets [], and braces { }**.

Example 1

Write an expression representing the phrase "add 4 and 3, then multiply the answer by 2."

Solution

You need to show that the addition should be done first, so put that part **inside grouping symbols**:

The expression should be $(4 + 3) \times 2$.

✓ Guided Practice

Write numeric expressions for these phrases:

1. Divide 4 by 8 then add 3.
2. Divide 4 by the sum of 8 and 3.
3. From 20, subtract the product of 8 and 2.
4. From 20, subtract 8 and multiply by 2.

Evaluate the following sums and differences:

- | | |
|-------------------|------------------------------|
| 5. $(3 - 2) - 5$ | 8. $(7 - 8) - (-3 - 8) - 11$ |
| 6. $3 - (2 - 5)$ | 9. $(5 - 9) - (3 - 10) - 2$ |
| 7. $6 - (11 + 7)$ | 10. $9 + (5 - 3) - 4$ |

Nested grouping symbols are when you have grouping symbols inside other grouping symbols.

When you see nested grouping symbols, you always start from the inside and work outwards.

Example 2

Evaluate $\{5 - [11 - (7 - 2)]\} + 34$.

Solution

Start from the inside and work outwards:

$$\begin{aligned}\{5 - [11 - (7 - 2)]\} + 34 &= \{5 - [11 - 5]\} + 34 \\ &= \{5 - 6\} + 34 \\ &= -1 + 34 \\ &= 33\end{aligned}$$

Guided Practice

Evaluate the following:

11. $8 + [10 + (6 - 9) + 7]$
12. $9 - \{[(-4) + 10] + 7\}$
13. $[(13 - 12) + 6] - (4 - 2)$
14. $14 - \{8 + [5 - (-2)]\} - 6$
15. $13 + [10 - (4 + 5)] - (11 + 8)$
16. $10 + \{[7 - (-2)] - (3 - 1)\} + (-14)$

There are Other Rules About What to Evaluate First

This order of operations is used by all mathematicians, so that every mathematician in the world evaluates expressions in the same way.

1. First calculate expressions within **grouping symbols** — working from the innermost grouping symbols to the outermost.
2. Then calculate expressions involving **exponents**.
3. Next do all **multiplication** and **division**, working from left to right. Multiplication and division have equal priority, so do them in the order they appear from left to right.
4. Lastly, do any **addition** or **subtraction**, again from left to right. Addition and subtraction have the same priority, so do them in the order they appear from left to right too.

Check it out

See Topic 1.3.1 for more about exponents.

Check it out

When you're simplifying expressions within grouping symbols, follow steps 2–4 (see Example 3).

Example 3

Simplify $\{4(10 - 3) + 3^2\} \times 5 - 11$.

Solution

Work through the expression bit by bit: $\{4(10 - 3) + 3^2\} \times 5 - 11$

$$= \{4 \times 7 + 3^2\} \times 5 - 11 \quad \text{innermost grouping symbols first}$$

Now you have to calculate everything inside the **remaining grouping symbols**:

$$\begin{aligned} &= \{4 \times 7 + 9\} \times 5 - 11 && \text{first work out the exponent} \\ &= \{28 + 9\} \times 5 - 11 && \text{then do the multiplication} \\ &= 37 \times 5 - 11 && \text{then do the addition} \end{aligned}$$

Now there are **no grouping symbols** left, so you can do the rest of the calculation:

$$\begin{aligned} &= 185 - 11 && \text{do the multiplication first} \\ &= \mathbf{174} && \text{and finally the subtraction} \end{aligned}$$

✓ Guided Practice

Evaluate the following:

17. $24 \div 8 - 2$

18. $3^2 - 4 \times 2$

19. $21 - \{-3[-5 \times 4 + 3^2] - 9 \times 2^3\} + 17$

20.
$$\frac{-3(3^2 - 4)}{3 + [12 \div [10 + 3(-4)]]}$$

21. $8 + \{10 \div [11 - 6] \times (-4)\}$

22. $[17 + \{(-3^3 + 4) \times 8 - 17\}] \div 8$

✓ Independent Practice

Evaluate the following.

1. $14 - [9 - 4 \times 2]$

2. $11 + (9 - 3) - (4 \div 2)$

3. $(-1) \times (7 - 10 + 12)$

4. $12 + 9 \times [(1 + 2) - (6 - 14)]$

5. $[(11 - 8) + (7 - 2)] \times 3 - 13$

6. $[(10 + 9 + 5) \div 2] - (6 - 12)$

Insert grouping symbols in each of the following statements so that each statement is true:

7. $12 + 4^2 \times 24 - 18 \div 3 = 44$

8. $20 + 3^2 - 14 - 12 \times 6 = 517$

9. $4 \times 3^3 - 2 \times 3^2 - 3 \times 4 + 6 = 1662$

Round Up

You really need to *learn* the order of operation rules. You'll be using them again and again in Algebra I so you might as well make sure you remember them right now.

Properties of Real Numbers

California Standards:

1.0: Students identify and use the arithmetic properties of subsets of integers and rational, irrational, and **real numbers**, including closure properties for the four basic arithmetic operations where applicable.

What it means for you:

You'll learn about the commutative, associative, and distributive laws of the real numbers.

Key words

- commutative
- associative
- distributive
- binary operation

In this Section you've already seen lots of real number properties — and now it's time for some more. These rules are really important because they tell you exactly how to deal with real numbers.

Commutative Laws — the Order Doesn't Matter

It doesn't matter which order you add two numbers in — the result is the same. This is called the **commutative property** (or law) **of addition**.

$$\text{For any } a, b \in \mathbb{R}: a + b = b + a$$

The same is true of multiplication — this is the **commutative property** (or law) **of multiplication**.

$$\text{For any } a, b \in \mathbb{R}: a \times b = b \times a$$

For example:

$$2 + 3 = 3 + 2 \text{ — commutative property of } \mathbf{addition}$$

$$2 \times 3 = 3 \times 2 \text{ — commutative property of } \mathbf{multiplication}$$

Associative Laws — You Can Group Numbers Any Way

Addition and multiplication are **binary operations** — you can only add or multiply **two** numbers at a time. So to add three numbers, for example, you add **two** of them first, and then add the **third** to the **result**.

However, it doesn't matter which two you add first:

$$\text{For any } a, b, c \in \mathbb{R}: (a + b) + c = a + (b + c)$$

Again, the same is true of **multiplication**:

$$\text{For any } a, b, c \in \mathbb{R}: (a \times b) \times c = a \times (b \times c)$$

These are called the **associative properties** (laws) of addition and multiplication.

For example:

$$(2 + 3) + 4 = 2 + (3 + 4) \text{ — associative property of } \mathbf{addition}$$

$$(2 \times 3) \times 4 = 2 \times (3 \times 4) \text{ — associative property of } \mathbf{multiplication}$$

Distributive Laws — Multiply Out Parentheses

The **distributive law of multiplication over addition** defines how multiplication and addition combine.

When you **multiply** by a sum in parentheses, it's the distributive law that you're using.

$$\text{For any } a, b, c \in \mathbb{R}: a \times (b + c) = (a \times b) + (a \times c)$$

So a factor outside parentheses multiplies every term inside.

Example 1

Expand: a) $3(x + y)$ b) $x(3 + y)$ c) $3[(x + y) + z]$

Solution

$$\text{a) } 3(x + y) = \mathbf{3x + 3y}$$

$$\begin{aligned} \text{b) } x(3 + y) &= x \cdot 3 + xy \\ &= \mathbf{3x + xy} \quad (\text{using the commutative law of multiplication}) \end{aligned}$$

$$\begin{aligned} \text{c) } 3[(x + y) + z] &= 3(x + y) + 3z \\ &= (3x + 3y) + 3z \\ &= \mathbf{3x + 3y + 3z} \end{aligned}$$

Check it out

In c), it's fine to leave the parentheses out. It doesn't matter if you do $(3x + 3y) + 3z$ or $3x + (3y + 3z)$ — the result is the same.

Independent Practice

Identify the property that makes each of the following statements true.

- $8 \times 3 = 3 \times 8$
- $2(x + 3) = 2x + 6$
- $9 + 2 = 2 + 9$
- $(9 + 2) + 1 = 9 + (2 + 1)$
- $4(p + 1) = 4p + 4$
- $(9 \times 4) \times 5 = 9 \times (4 \times 5)$

Expand:

- $a(b + c)$
- $6(t + 5)$
- $12t(s - r)$

Round Up

*The thing with these laws is that you've been using them for years without thinking about them, so it might seem strange to be taught them now. But when you **rearrange equations**, you're using these laws. In Algebra I you can use these laws to justify anything you do to an equation.*

More on Multiplication

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

What it means for you:

You'll learn about multiplicative opposites, and numbers that don't change a value when you multiply by them.

Key words

- inverse
- identity
- reciprocal

Check it out

See Topic 1.2.5 for more on additive inverses.

Check it out

A minus sign outside parentheses changes the sign of everything inside the parentheses.

You already dealt with changing signs during multiplication in Topic 1.2.4. This Topic is all about *opposites in multiplication*, and numbers that *don't change a value if you multiply by them*.

Multiply by -1 to Find the Additive Inverse or Opposite

Every real number has an **additive inverse** — a number that will give zero when added to that number.

So for every number m , there is an opposite number (or negative), $-m$.

In particular, the number 1 has the additive inverse -1 .

And if you multiply any number by -1 , you get that number's inverse.

$$\text{For any } m \in \mathbb{R}: -m = -1 \times m = m \times -1$$

So the opposite of 4 is $-1 \times 4 = -4$. And the opposite of x is $-1 \times x = -x$.

Example 1

Find the opposite of $(2a + 3)$.

Solution

The question is asking you to find $-(2a + 3)$.

Find the opposite by multiplying by -1 :

$$\begin{aligned} -(2a + 3) &= -1 \times (2a + 3) \\ &= [(-1) \times (2a)] + [(-1) \times (3)] \\ &= [(-1 \times 2) \times a] + (-3) \\ &= -2a + (-3) \\ &= \mathbf{-2a - 3} \end{aligned}$$

✓ Guided Practice

Find the opposite of each of these expressions.

- | | |
|------------|-------------|
| 1. t | 4. $a - 2$ |
| 2. 3 | 5. $-x + 1$ |
| 3. $t + 3$ | 6. $-y - 4$ |

Anything Multiplied by 0 is 0

Zero is a special number — it's the additive identity.

This means that for any number m , $m + 0 = m = 0 + m$.

But zero has another useful property as well...

The product of any real number and 0 equals 0.

$$\text{For any } m \in \mathbb{R}: m \times 0 = 0 = 0 \times m$$

One result of this is that **zero doesn't have a reciprocal**, since there is no number that you can possibly multiply zero by in order to get 1 — the multiplicative identity.

What this means in practice is that you **can't divide by zero**.

Check it out

See Topic 1.2.5 for the definition of division.

You Can't Multiply Two Nonzero Numbers and Get Zero

If nonzero real numbers are multiplied together, the result is never zero.

The product of two (or more) nonzero real numbers cannot be zero.

$$\text{If } x, y \in \mathbb{R}, \text{ and } xy = 0, \text{ then either } x = 0 \text{ or } y = 0 \\ \text{(or both } x = 0 \text{ and } y = 0).$$

Check it out

See Topic 7.1.1 for more examples of solving equations using this rule.

This has practical uses when it comes to solving equations.

Example 2

Find two possible solutions to the equation $x(x - 1) = 0$.

Solution

There are two expressions multiplied together to give zero.

Either one or the other must equal 0, so $x = 0$ or $x - 1 = 0$

— that is, **$x = 0$ or $x = 1$** .

(You need to write “or,” since x can't be both 0 and 1 at the same time.)

Independent Practice

Find the opposite.

1. -4

4. $t - 6$

2. $-a$

5. $-b + 8$

3. $g + 5$

Solve each equation.

6. $x + 1 = 0$

8. $y(y + 2) = 0$

7. $y - 4 = 0$

9. $t(t - 3) = 0$

Round Up

These rules might just sound like common sense — but it's important to write statements in formal math-speak to prove that they're true. You'll use these rules throughout Algebra I.

The Real Number Axioms

California Standards:

1.1: Students use properties of numbers to demonstrate whether assertions are true or false.

What it means for you:

You'll see that the properties of real numbers that you learned in this section are useful to prove whether math statements are true.

Key words

- theorem
- axiom
- postulate
- closure
- identity
- inverse
- commutative
- associative
- distributive

Most of what has been covered in this Section has been about the *axioms* (or *postulates*) of the real number system. This Topic gives a summary of the axioms, and shows how you can use them.

Axioms are Fundamental Assumptions

A **theorem** is a statement that **can be proved**.

An **axiom**, on the other hand, is a fundamental **assumption** — a statement that is **accepted as true without having to be proved**.

You can use these axioms to **justify solution steps** when **simplifying mathematical expressions, proving theorems, solving equations, and supporting mathematical arguments**.

You have to know all the axioms in this section, along with their names. Here's a summary of them all together:

<u>Property Name</u>	<u>Addition</u>	<u>Multiplication</u>
Closure Property:	$a + b$ is a real number $3 + 5 = 8 \in \mathbb{R}$	$a \times b$ is a real number $3 \times 5 = 15 \in \mathbb{R}$
Identity Property:	$a + 0 = a = 0 + a$ $3 + 0 = 3 = 0 + 3$	$a \times 1 = a = 1 \times a$ $3 \times 1 = 3 = 1 \times 3$
Inverse Property:	$a + (-a) = 0 = -a + a$ $3 + (-3) = 0 = (-3) + 3$	$a \times a^{-1} = 1 = a^{-1} \times a$ $3 \times 3^{-1} = 1 = 3^{-1} \times 3$
Commutative Property:	$a + b = b + a$ $3 + 5 = 5 + 3$	$a \times b = b \times a$ $3 \times 5 = 5 \times 3$
Associative Property:	$(a + b) + c = a + (b + c)$ $(3 + 5) + 6 = 3 + (5 + 6)$	$(ab)c = a(bc)$ $(3 \times 5) \times 6 = 3 \times (5 \times 6)$
Distributive Property of Multiplication over Addition:		$a(b + c) = ab + ac$ $3(5 + 6) = 3 \times 5 + 3 \times 6$ and $(b + c)a = ba + ca$ $(5 + 6)3 = 5 \times 3 + 6 \times 3$

Use the Axioms to Justify Steps in Proofs

These axioms can be used to justify steps in a mathematical proof.

Check it out

Any of the properties covered in the section as well as the axioms on the previous page can be used to justify steps.

Example 1

Show that $(x + y) - x = y$. Justify your steps.

Solution

$$\begin{aligned}(x + y) - x &= (y + x) + (-x) && \text{Commutative property of addition,} \\ &&& \text{and the definition of subtraction} \\ &= y + [x + (-x)] && \text{Associative property of addition} \\ &= y + 0 && \text{Definition of an additive inverse} \\ &= y && \text{Identity property of addition}\end{aligned}$$

Check it out

You don't usually need to write down how to justify each step — but you should know how to.

Example 2

Show that $(x + y)(x - y) = x^2 - y^2$. Justify your steps.

Solution

$$\begin{aligned}(x + y)(x - y) &= (x + y)(x + (-y)) && \text{Definition of subtraction} \\ &= (x + y)x + (x + y)(-y) && \text{Distributive law} \\ &= x^2 + yx + x(-y) + y(-y) && \text{Distributive law} \\ &= x^2 + xy + x(-1 \cdot y) + y(-1 \cdot y) && \text{Commutative law of } \times, \\ &&& \text{and multiplication property} \\ &&& \text{of } -1 \\ &= x^2 + xy + [x(-1)]y + [y(-1)]y && \text{Associative law of } \times \\ &= x^2 + xy + (-xy) + (-1)(y^2) && \text{Commutative law of } \times, \\ &&& \text{and multiplication property} \\ &&& \text{of } -1 \\ &= x^2 + 0 + (-y^2) && \text{Inverse property of } + \\ &= x^2 - y^2 && \text{Identity property of } +, \text{ and} \\ &&& \text{definition of subtraction}\end{aligned}$$

✓ Independent Practice

State the real number property that justifies each statement below:

- $3 + 5$ is a real number.
- 7×2 is a real number.
- $m + c = c + m$ for any real numbers m and c .
- $mc = cm$ for any real numbers m and c .
- $12 \times (7 \times 4) = (12 \times 7) \times 4$
- $5(m - v) = 5m - 5v$
- $8 + (7 + 4) = (8 + 7) + 4$
- $m^{-1} \times m = m \times m^{-1} = 1$
- $5 + 0 = 0 + 5 = 5$
- $10 \times 1 = 1 \times 10 = 10$

11. The following is a proof showing that $0c = 0$, for any real c . Fill in the missing properties to support each step in the proof.

$$\begin{aligned}0c &= 0c + 0 \\ &= 0c + [c + (-c)] && \text{Inverse property of } + \\ &= [0c + c] + (-c) \\ &= [0c + 1c] + (-c) && \text{Identity property of } \times \\ &= [0 + 1]c + (-c) \\ &= 1c + (-c) \\ &= c + (-c) && \text{Identity property of } \times \\ &= 0\end{aligned}$$

12. Given real numbers m , c , and v , $m(c - v) = mc - mv$. Fill in the missing properties to support each step in the proof.

$$\begin{aligned}m(c - v) &= m[c + (-v)] \\ &= mc + m(-v) && \text{Distributive property of } \times \text{ over } + \\ &= mc + m(-1 \times v) \\ &= mc + [m \cdot (-1)] \times v && \text{Associative property of } \times \\ &= mc + [(-1) \cdot m] \times v \\ &= mc + (-1)(mv) && \text{Associative property of } \times \\ &= mc + (-mv) \\ &= mc - mv && \text{Definition of subtraction}\end{aligned}$$

Round Up

*This Section is full of **rules** and **properties**. These aren't just useless lists of abstract rules, though — you'll be using them throughout Algebra I. You've been using most of these rules in previous grades without realizing it — but now you know all the proper names for them too. You'll sometimes have to state which rules you're using when you're doing math problems.*

Section 1.3 Exponent Laws

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

What it means for you:

You'll learn about the rules of exponents.

Key words:

- exponent
- base
- power
- product
- quotient

Exponents have a whole set of rules to make sure that all mathematicians deal with them in the same way. There are a lot of rules written out in this Topic, so take care.

Powers are Repeated Multiplications

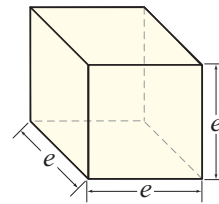
A **power** is a multiplication in which all the factors are the same.

For example, $m^2 = m \times m$ and $m^3 = m \times m \times m$ are both powers of m .

In this kind of expression, “ m ” is called the **base** and the “2” or “3” is called the **exponent**.

Example 1

- Find the volume of the cube shown. Write your answer as a power of e .
- If the edges of the cube are 4 cm long, what is the volume?



Solution

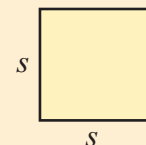
- $V = e \times e \times e = e^3$
- $V = e^3 = (4 \text{ cm})^3$
 $= 4^3 \text{ cm}^3$
 $= 64 \text{ cm}^3$

Guided Practice

Expand each expression and evaluate.

- 2^3
- 3^2
- $5^2 \times 3^2$
- $2^4 y^3$

- Find the area, A , of the square shown. Write your answer as a power of s .



- If the sides of the square are 7 inches long, what is the area?

- Find the volume of a cube if the edges are 2 feet long. (Volume $V = e^3$, where e is the edge length.)

There are Lots of Rules of Exponents

1) If you multiply m^2 by m^3 , you get m^5 , since:

$$\begin{aligned}m^2 \times m^3 &= (m \times m) \times (m \times m \times m) \\ &= m \times m \times m \times m \times m \\ &= m^5\end{aligned}$$

The exponent of the product is the same as the exponents of the factors added together. This result always holds — to **multiply** powers with the same base, you simply **add the exponents**.

$$m^a \times m^b = m^{a+b}$$

2) In a similar way, to **divide** powers, you **subtract the exponents**.

$$m^a \div m^b = m^{a-b}$$

3) When you raise **a power to a power**, you multiply the exponents — for example, $(m^3)^2 = m^3 \times m^3 = m^6$.

$$(m^a)^b = m^{ab}$$

4) Raising a **product** or **quotient** to a power is the same as raising each of its elements to that power. For example:

$$\begin{aligned}(mb)^3 &= mb \times mb \times mb \\ &= (m \times b) \times (m \times b) \times (m \times b) \\ &= m \times m \times m \times b \times b \times b = m^3 b^3.\end{aligned}$$

$$(mb)^a = m^a b^a$$

$$\left(\frac{m}{b}\right)^a = \frac{m^a}{b^a}$$

5) Using rule 1 above: $m^a \times m^0 = m^{a+0} = m^a$. So m^0 equals 1.

$$m^0 = 1$$

6) It's also possible to make sense of a negative exponent:

$$m^a \times m^{-a} = m^{a-a} = m^0 = 1$$

(using rules 1 and 5 above)

So the reciprocal of m^a is m^{-a} .

$$(m^a)^{-1} = m^{-a} = \frac{1}{m^a}$$

7) And taking a root can be written using a fractional power.

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

These rules **always work**, unless the base is 0.

The exponents and the bases can be positive, negative, whole numbers, or fractions. The only exception is you cannot raise zero to a negative exponent — zero does not have a reciprocal.

✓ Independent Practice

In exercises 1–6, write each expression using exponents.

1. $2 \times 2 \times 2 \times 2$

2. $a \times a \times a \times 4$

3. $2 \times k \times 2 \times 2 \times k$

4. $4 \times 3 \times 3 \times 4 \times p \times 3 \times 3 \times p \times 4$

5. $a \times b \times a \times b$

6. $5 \times l \times 3 \times 5 \times 5 \times l$

7. Show that $\frac{k^6}{k^4} = k^2$.

Simplify the expressions in exercises 8–25 using rules of exponents.

8. 17^0

9. 2^{-3}

10. $2^2 \cdot 2^3$

11. $\frac{3^6}{3^4}$

12. $(2^3)^2 \cdot 2^2$

13. $\frac{2^3 \cdot 3^4}{3^7}$

14. $\frac{(3^2)^2}{3^3}$

15. $(x^4 \div x^2) \cdot x^3$

16. $(x^2)^3 \div x^4$

17. $\frac{x^3 \cdot x^5}{(ax)^2}$

18. $\frac{(x^3)^{-3}}{x^{-4}} \cdot x^5$

19. $(2x^{-2})^3 \cdot 4x^2$

20. $3x^0y^{-2}$

21. $(3x)^0xy^{-2}$

22. $5x^{-1} \times 6(xy)^0$

23. $\frac{(4x)^2y}{2x}$

24. $\frac{(2x^3)^2y}{y^{-2}}$

25. $\frac{(3^2x^5y^3)^{-2}}{x^4y^{-6}}$

26. An average baseball has a radius, r , of 1.45 inches. Find the volume, V , of a baseball in cubic inches. ($V = \frac{4}{3}\pi r^3$)

27. The kinetic energy of a ball (in joules) is given by $E = \frac{1}{2}mv^2$, where m is the ball's mass (in kilograms) and v is its velocity (in meters per second). If a ball weighs 1 kilogram and is traveling at 10 meters per second, what is its kinetic energy in joules?

28. The speed of a ball (in meters per second) accelerating from rest is given by $v = \frac{1}{2}at^2$, where a is its acceleration (in meters per second squared) and t is its time of flight (in seconds). Calculate the speed of a ball in meters per second after 5 seconds of flight if it is accelerating at 5 meters per second squared.

Round Up

*That's a lot of rules, but don't worry — you'll get plenty of practice using them later in the program. Exponents often turn up when you're dealing with **area** and **volume**. The next Topic will deal just with **square roots**, which is a special case of Rule 7 from the previous page.*

Square Roots

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

What it means for you:

You'll look more closely at the rules of square roots.

Key words:

- square root
- radical
- radicand
- principal square root
- minor square root

Check it out:

Technically, the square root of a number p can be written as $\sqrt[2]{p}$, but you don't usually write the 2.

In the last Topic you learned about all the *exponent rules* — this Topic will look more closely at one rule in particular. *Square roots* are the type of root that you'll come across most often in math problems — so it's really important that you know how to deal with them.

Another Name for the Root Sign is the Radical Sign

The **square root** of p is written \sqrt{p} .

If you multiply \sqrt{p} by itself, you get p — so $\sqrt{p} \times \sqrt{p} = p$.

Multiplying \sqrt{p} by itself means you **square** it.

The **n th root** of p is written $\sqrt[n]{p}$.

If you raise $\sqrt[n]{p}$ to the power n , you get p — so $(\sqrt[n]{p})^n = p$.

The symbol $\sqrt{\quad}$ is called the **radical** sign and **shows the nonnegative root**

if more than one root exists. In the expression $\sqrt[n]{p}$ (the n th root of p), p is called the **radicand**.

The square root of a number p is also written $p^{\frac{1}{2}}$.

You can show this using the rules of exponents: $\left[p^{\frac{1}{2}}\right]^2 = p^{\frac{2}{2}} = p^1 = p$

For any real number $p > 0$, the **square root** is written as \sqrt{p} (or $p^{\frac{1}{2}}$).

If $r = \sqrt{p}$, then $r^2 = p$ and $(-r)^2 = p$.

r is called the **principal square root** of p and $-r$ is called the **minor square root** of p .

✓ Guided Practice

Complete the following.

1. The radicand of $\sqrt[3]{8}$ is
2. The 6th root of t is written in radical notation.
3. $\sqrt{9} \times \dots = 9$
4. $b^{\frac{1}{2}} = \dots$ in radical notation.

Positive Numbers Have Two Square Roots

In practice, this means that every positive number has **two square roots** — a **positive** one (the **principal square root**) and a **negative** one (the **minor square root**).

Example 1

Find the square roots of the following numbers:

- a) 100 b) n^2

Solution

a) $\sqrt{100} = 10$, so the principal square root is **10**, and the minor square root is **-10**.

b) $\sqrt{n^2} = |n|$, so the principal square root is **$|n|$** , and the minor square root is **$-|n|$** .

The principal square root of n is written as \sqrt{n} . The minor square root of n is written as $-\sqrt{n}$. To indicate both square roots you can write $\pm\sqrt{n}$.

✓ Guided Practice

Find the principal square root and minor square root of these numbers:

5. 4 6. 100 7. 81

Use the “ \pm ” symbol to give the principal and minor square root of the following numbers:

8. 9 9. 16 10. 144

11. 35^2 12. x^2 13. 81

14. t^2 15. 9×9 16. $(st)^2$

Evaluate the following, giving the principal and minor roots:

17. $4^{\frac{1}{2}}$ 18. $121^{\frac{1}{2}}$

Algebraic Expressions Also Have Square Roots

You can also take the square root of an **algebraic expression**.

Example 2

Find the square root of $(x + 1)^2$.

Solution

$\sqrt{(x + 1)^2} = |x + 1|$, so the principal square root is $|x + 1|$ and the minor square root is $-|x + 1|$.

Guided Practice

Give the principal and minor square root of each of the following expressions.

19. $t \times t$

20. $t^2 \times t^2$

21. $a^2 \times a^2$

22. $(a + b) \times (a + b)$

23. $t(a + b) \times t(a + b)$

24. $(a + b)^2$

25. $(t + 1)^2$

26. $[t(a + b)]^2$

27. $[2(a + b)]^2$

Independent Practice

1. Is this statement true or false? “The radicand of $\sqrt[5]{32}$ is 5.”

Evaluate the following.

2. $\sqrt{64}$

3. $(49)^{\frac{1}{2}}$

4. $\sqrt{a^2}$

5. $\sqrt{25}$

6. $\sqrt{12^2}$

7. $\sqrt{j \times j}$

Find the square roots of the following.

8. $(a^2)^2$

9. $(k - 1)^2$

10. $(m + n)^2$

11. $(m^2 + n^2)^2$

12. $(2pq)^2$

13. $[(a + b) \times (c + d)]^2$

Round Up

Remember that when you take the square root of a positive number, you always have **two possible answers** — a positive one and a negative one. You can give both answers neatly using the \pm sign.

Properties of Roots

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, **taking a root**, and **raising to a fractional power**. They understand and use the rules of exponents.

What it means for you:

You'll find out how to multiply and divide square roots.

Key words:

- square root
- radical
- radicand
- numerator
- denominator

Check it out:

Remember, \sqrt{mc} means the principal (nonnegative) square root of mc .

In the last Topic you saw that positive numbers have *two square roots*. This Topic's all about how to *multiply and divide square roots*, which is really important when you're evaluating expressions involving *more than one root sign*.

Numbers Can Be Split into Factors

A **factor of a number** is a number that divides into it **without a remainder** — for example, 1, 2, 5, and 10 are factors of 10.

To **factor a number** or expression means to write it as a **product of its factors** — for example, $10 = 2 \times 5$.

Factoring is a useful way of **simplifying square roots** — as you'll see in the rest of this lesson.

There's a Multiplicative Property of Square Roots

$$\sqrt{mc} = \sqrt{m} \cdot \sqrt{c}$$

This means that to make finding a square root easier, you can try to **factor** the radicand first.

Example 1

Find the following:

a) $\sqrt{400}$ b) $\sqrt{8}$ c) $\sqrt{(x+1)^2(3x-1)^2}$

Solution

$$\begin{aligned} \text{a) } \sqrt{400} &= \sqrt{4 \times 100} & \text{b) } \sqrt{8} &= \sqrt{4 \times 2} \\ &= \sqrt{4} \times \sqrt{100} & &= \sqrt{4} \times \sqrt{2} \\ &= \sqrt{2^2} \times \sqrt{10^2} & &= 2 \times \sqrt{2} \\ &= 2 \times 10 & &= \mathbf{2\sqrt{2}} \\ &= \mathbf{20} \end{aligned}$$

The same technique can work if you have an algebraic expression. You need to find factors that are squares of other expressions:

$$\text{c) } \sqrt{(x+1)^2(3x-1)^2} = \sqrt{(x+1)^2} \sqrt{(3x-1)^2} = \mathbf{[(x+1)(3x-1)]}$$

✓ Guided Practice

Simplify the following square roots.

1. $\sqrt{900}$

6. $\sqrt{4x^2}$

2. $\sqrt{225}$

7. $\sqrt{81t^2}$

3. $\sqrt{20}$

8. $\sqrt{4(t+2)^2}$

4. $\sqrt{200}$

9. $\sqrt{36(j-3)^2}$

5. $\sqrt{32}$

10. $\sqrt{64(k+4)^2}$

There's a Division Property of Square Roots Too

$$\sqrt{\frac{m}{c}} = \frac{\sqrt{m}}{\sqrt{c}}$$

Again, the idea is to look for any factors in the numerator or denominator that are squares.

Example 2

Find the following:

a) $\sqrt{\frac{49}{225}}$

b) $\sqrt{\frac{3}{16}}$

c) $\sqrt{\frac{(2x+1)^2}{(3x-1)^2}}, \quad x \neq \frac{1}{3}$

Solution

a) $\sqrt{\frac{49}{225}} = \frac{\sqrt{49}}{\sqrt{225}} = \frac{\sqrt{7^2}}{\sqrt{15^2}} = \frac{7}{15}$

b) $\sqrt{\frac{3}{16}} = \frac{\sqrt{3}}{\sqrt{16}} = \frac{\sqrt{3}}{\sqrt{4^2}} = \frac{\sqrt{3}}{4}$

c) $\sqrt{\frac{(2x+1)^2}{(3x-1)^2}} = \frac{\sqrt{(2x+1)^2}}{\sqrt{(3x-1)^2}} = \frac{|2x+1|}{|3x-1|}$

Check it out:

In Example 1 c), the numerator and the denominator are both squares.

✓ Guided Practice

Find the following.

11. $\sqrt{\frac{16}{4}}$

12. $\sqrt{\frac{25}{9}}$

13. $\sqrt{\frac{125}{16}}$

14. $\sqrt{\frac{50}{4}}$

15. $\sqrt{\frac{x^2}{36}}$

16. $\sqrt{\frac{200}{x^2}}, x \neq 0$

17. $\sqrt{\frac{242}{a^2}}, a \neq 0$

18. $\sqrt{\frac{(d-1)^2}{(f+1)^2}}, f \neq -1$

19. $\sqrt{\frac{(a+b)^2}{(c+d)^2}}, c+d \neq 0$

✓ Independent Practice

Find the following.

1. $\sqrt{49 \times 4}$

2. $\sqrt{25m^2}$

3. $\sqrt{\frac{64}{9}}$

4. $\sqrt{\frac{121}{144}}$

5. $\sqrt{\frac{t^2}{81}}$

6. $\sqrt{48t^2}$

7. $\sqrt{72x^2}$

8. $\sqrt{\frac{8}{m^2}}, m \neq 0$

9. $\sqrt{27y^2}$

10. $\sqrt{\frac{300}{t^2}}, t \neq 0$

11. $\sqrt{\frac{x^2y^2}{81}}$

12. $\sqrt{\frac{36y^2}{49}}$

13. $\sqrt{\frac{100t^2}{16a^2}}, a \neq 0$

14. $\sqrt{\frac{16x^2}{y^2}}, y \neq 0$

15. $\sqrt{\frac{8x^2}{9y^2}}, y \neq 0$

Round Up

Root signs are really tricky, so it's a good idea to get rid of them whenever you can. When you *multiply or divide roots*, you're left with simpler expressions, which makes math problems a lot easier.

Equivalent Fractions

California Standards:

2.0: Students understand and use such operations as taking the opposite, **finding the reciprocal**, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

What it means for you:

You'll simplify fractions to their lowest terms.

Key words:

- equivalent
- numerator
- denominator
- greatest common factor
- simplify

Here's another Algebra I Topic that you've seen in earlier grades. You've used *fractions* a lot before, but in Algebra I you'll treat them more *formally*. This Topic goes over stuff on *simplifying fractions* that should feel quite familiar to you.

Fractions Have a Numerator and a Denominator

A **fraction** is any number expressed as one **integer** divided by another **integer**.

Fractions are written in the form $\frac{p}{q}$, where p and q are integers, and $q \neq 0$.

The top number (p) is called the **numerator** and the bottom number (q) is the **denominator**. A fraction with a denominator of zero is undefined, because you can't divide by zero.

Equivalent Fractions Have the Same Value

Equivalent fractions are fractions of the same value — for example,

$$\frac{2}{3} \text{ and } \frac{4}{6}, \frac{1}{2} \text{ and } \frac{4}{8}.$$

To simplify $\frac{4}{6}$ to $\frac{2}{3}$, you can rewrite the numerator and denominator as

products of factors (**factor** them). You can then **cancel any common factors** by dividing both the numerator and denominator by those factors to produce an equivalent fraction.

Example 1

Convert $\frac{4}{6}$ to $\frac{2}{3}$.

Solution

First factor the numerator and denominator: $\frac{4}{6} = \frac{2 \cdot 2}{2 \cdot 3}$.

You can cancel factors that are common to both the numerator and denominator, so in this case, parts of the fractions cancel, leaving:

$$\frac{\cancel{2} \cdot 2}{\cancel{2} \cdot 3} = \frac{2}{3}$$

✓ Guided Practice

1. Identify the numerator and the denominator in the fraction $\frac{9}{10}$.

In Exercises 2-4, convert the fractions to $\frac{3}{4}$.

2. $\frac{12}{16}$

3. $\frac{9}{12}$

4. $\frac{21}{28}$

In exercises 5-7, convert the fractions to $\frac{5}{8}$.

5. $\frac{25}{40}$

6. $\frac{15}{24}$

7. $\frac{40}{64}$

8. Show that $\frac{3}{5}$ and $\frac{9}{15}$ are equivalent fractions.

9. Show that $\frac{5}{10}$ and $\frac{3}{6}$ are equivalent fractions.

You Can Create Equivalent Fractions Too

The **greatest common factor** of two numbers a and b is the **largest** possible number that will divide exactly into both a and b .

To **simplify** a fraction (or reduce it to its lowest terms) is to convert it to an equivalent fraction in which the numerator and denominator have a **greatest common factor of 1**.

Example 2

Reduce $\frac{56}{64}$ to its lowest terms.

Solution

$$\frac{56}{64} = \frac{7 \cdot 8}{8 \cdot 8}$$

Factor the numerator and denominator

$$\frac{7 \cdot \cancel{8}}{8 \cdot \cancel{8}} = \frac{7}{8}$$

Cancel common factors from the numerator and denominator

The **greatest common factor** (GCF) of 7 and 8 is 1, so this is the simplest form of $\frac{56}{64}$.

Similarly, you can produce an equivalent fraction by **multiplying** both the numerator and denominator by the same number.

✓ Guided Practice

Complete these statements.

10. The largest possible number that will divide exactly into two numbers a and b is called the of a and b .

11. A fraction is expressed in its lowest terms if the numerator and denominator have a GCF of

Find the greatest common factor of each of these pairs.

12. 81 and 90 13. 56 and 77 14. 42 and 54 15. 13 and 19

Simplify these fractions.

16. $\frac{4}{12}$ 17. $\frac{12}{14}$ 18. $\frac{30}{33}$ 19. $\frac{9}{24}$

✓ Independent Practice

1. Identify the numerator in the fraction $\frac{11}{13}$.

2. Identify the denominator in the fraction $\frac{14}{15}$.

In exercises 3–5, show how to simplify each fraction to $\frac{2}{3}$.

3. $\frac{10}{15}$ 4. $\frac{18}{27}$ 5. $\frac{8}{12}$

6. Show that $\frac{10}{16}$ and $\frac{5}{8}$ are equivalent fractions.

Find the greatest common factor of each of these pairs of numbers.

7. 15 and 20 8. 21 and 33 9. 26 and 39

Simplify these fractions.

10. $\frac{12}{15}$ 11. $\frac{12}{20}$ 12. $\frac{21}{39}$
13. $\frac{44}{48}$ 14. $\frac{81}{90}$ 15. $\frac{56}{77}$

Round Up

Hopefully you recognized a lot of the stuff in this Topic from earlier grades. In the next couple of Topics you'll go over *multiplying, dividing, adding, and subtracting fractions* — and you'll need to be happy with simplifying fractions each time.

Multiplying and Dividing Fractions

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

What it means for you:

You'll multiply and divide fractions, then simplify them.

Key words:

- numerator
- denominator
- reciprocal

You did lots of work on *multiplying and dividing fractions* in grade 7. This Topic is mainly a reminder of those techniques, because you're going to be using them a lot in later math problems in Algebra I.

Multiply Fractions by Multiplying the Top and Bottom

To multiply two fractions, find the product of the numerators and divide that by the product of the denominators.

Given any numbers $a, b, c,$ and $d \in \mathbb{R}$ ($b \neq 0, d \neq 0$):

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{ac}{bd}$$

Example 1

a) Multiply $\frac{2}{5}$ and $\frac{7}{3}$.

b) Multiply $\frac{3}{4}$ and $\frac{9}{2}$.

Solution

a) $\frac{2}{5} \cdot \frac{7}{3} = \frac{2 \cdot 7}{5 \cdot 3} = \frac{14}{15}$

b) $\frac{3}{4} \cdot \frac{9}{2} = \frac{3 \cdot 9}{4 \cdot 2} = \frac{27}{8}$

✓ Guided Practice

Find each of these products.

1. $\frac{2}{3} \cdot \frac{4}{5}$

2. $\frac{3}{5} \cdot \frac{1}{2}$

3. $\frac{6}{7} \cdot \frac{3}{7}$

4. $\frac{3}{10} \cdot \frac{1}{8}$

5. $\frac{7}{8} \cdot \frac{3}{4}$

6. $\frac{5}{8} \cdot \frac{5}{8}$

7. $\frac{8}{11} \cdot \frac{3}{5}$

8. $\frac{9}{10} \cdot \frac{3}{5}$

9. $\frac{2}{7} \cdot \frac{5}{11}$

10. $\frac{3}{8} \cdot \frac{15}{2}$

11. $\frac{5}{9} \cdot \frac{4}{7}$

12. $\frac{8}{5} \cdot \frac{7}{3}$

Always Give Solutions in the Simplest Form

You should always give your answer in its simplest form — so with more complicated examples, **factor** the numerators and denominators and **cancel** common factors. It'll save time if you do this **before** you compute the products.

Example 2

Multiply and simplify $\frac{56}{64} \cdot \frac{8}{28}$.

Solution

$$\frac{56}{64} \cdot \frac{8}{28} = \frac{7 \cdot 8}{8 \cdot 8} \cdot \frac{8}{7 \cdot 4}$$

Factor the numerators and denominators

$$\frac{\cancel{7} \cdot \cancel{8}}{\cancel{8} \cdot \cancel{8}} \cdot \frac{\cancel{8}}{\cancel{7} \cdot 4} = \frac{1}{4}$$

Cancel all the common factors

Check it out:

If there are two fractions multiplied together, you can cancel a factor from anywhere in the numerator with a matching factor in the denominator. They don't have to have come from the same fraction.

Guided Practice

Multiply and simplify these expressions.

13. $\frac{3}{5} \cdot \frac{7}{12}$

14. $\frac{8}{21} \cdot \frac{3}{16}$

15. $\frac{14}{25} \cdot \frac{40}{63}$

16. $\frac{4}{9} \cdot \frac{36}{37}$

17. $\frac{22}{15} \cdot \frac{75}{26}$

18. $\frac{36}{55} \cdot \frac{11}{12}$

19. $\frac{40}{15} \cdot \frac{24}{64}$

20. $\frac{81}{90} \cdot \frac{10}{90}$

21. $\frac{25}{21} \cdot \frac{28}{40}$

Divide Fractions by Multiplying by the Reciprocal

The reciprocal of a fraction $\frac{c}{d}$ is $\frac{d}{c}$, since $\frac{c}{d} \cdot \frac{d}{c} = 1$.

To divide by a fraction, you **multiply by its reciprocal**.

Given any nonzero numbers a , b , c , and $d \in \mathbb{R}$:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c} = \frac{ad}{bc}$$

Example 3Divide $\frac{72}{96}$ by $\frac{9}{144}$.**Solution**

$$\frac{72}{96} \div \frac{9}{144} = \frac{72}{96} \cdot \frac{144}{9}$$

Rewrite as a multiplication by the reciprocal

$$\frac{72}{96} \cdot \frac{144}{9} = \frac{8 \cdot 9}{12 \cdot 8} \cdot \frac{12 \cdot 12}{9}$$

Factor the numerators and denominators

$$\frac{\cancel{8} \cdot \cancel{9} \cdot \cancel{12} \cdot 12}{\cancel{12} \cdot \cancel{8} \cdot \cancel{9}} = 12$$

Cancel all the common factors

Guided Practice

In Exercises 1–6, divide and simplify.

22. $\frac{4}{9} \div \frac{16}{15}$

23. $\frac{5}{21} \div \frac{25}{14}$

24. $\frac{34}{39} \div \frac{42}{13}$

25. $\frac{48}{7} \div \frac{16}{35}$

26. $\frac{13}{27} \div \frac{2}{63}$

27. $\frac{15}{56} \div \frac{3}{32}$

28. $\frac{24}{32} \div \frac{9}{56}$

29. $\frac{40}{72} \div \frac{24}{36}$

30. $\frac{15}{70} \div \frac{36}{56}$

Independent Practice

Evaluate the following. Simplify your answer where appropriate.

1. $\frac{42}{4} \cdot \frac{10}{28}$

2. $\frac{18}{20} \cdot \frac{16}{24}$

3. $\frac{14}{12} \cdot \frac{2}{21}$

4. $\frac{24}{6} \cdot \frac{30}{8}$

5. $\frac{30}{50} \div \frac{45}{55}$

6. $\frac{18}{12} \div \frac{8}{6}$

7. $\frac{21}{22} \div \frac{21}{14}$

8. $\frac{21}{32} \div \frac{24}{36}$

9. $\frac{35}{14} \div \frac{21}{44}$

10. $\frac{36}{24} \div \frac{28}{48}$

11. $\frac{64}{72} \cdot \frac{88}{40}$

12. $\frac{12}{50} \cdot \frac{88}{44}$

Round Up

Unless you're told otherwise, you should always *cancel fraction solutions* down to the most simple form. In the next Topic you'll look at adding and subtracting fractions, which is a bit tougher.

Adding and Subtracting Fractions

California Standards:

2.0: Students understand and use such operations as taking the opposite, **finding the reciprocal**, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

What it means for you:

You'll add and subtract fractions with the same denominator, then fractions with different denominators.

Key words:

- numerator
- denominator
- least common multiple
- prime factor

You dealt with multiplying and dividing fractions in Topic 1.3.5. This Topic deals with **adding** and **subtracting**, which is a little bit harder if the **denominators are different** in each of the fractions.

+ and – with Fractions with the Same Denominator

To find the sum (or the difference) of two fractions with the **same denominator**, just add (or subtract) the **numerators**, then divide by the common denominator.

Example 1

a) Calculate $\frac{1}{7} + \frac{5}{7}$.

b) Calculate $\frac{5}{7} - \frac{1}{7}$.

Solution

a) Add the numerators and divide the answer by the common denominator, 7:

$$\frac{1}{7} + \frac{5}{7} = \frac{1+5}{7} = \frac{6}{7}$$

b) Subtract the second numerator from the first and divide the answer by the common denominator, 7:

$$\frac{5}{7} - \frac{1}{7} = \frac{5-1}{7} = \frac{4}{7}$$

✓ Guided Practice

Perform the indicated operations and simplify each expression in exercises 1–9.

1. $\frac{5}{6} + \frac{7}{6}$

2. $\frac{2}{15} + \frac{8}{15}$

3. $\frac{13}{18} + \frac{11}{18}$

4. $\frac{15}{32} + \frac{21}{32}$

5. $\frac{12}{27} + \frac{11}{27}$

6. $\frac{24}{49} - \frac{11}{49}$

7. $\frac{53}{32} - \frac{15}{32}$

8. $\frac{32}{45} - \frac{37}{45}$

9. $\frac{21}{16} - \frac{15}{16}$

Adding Fractions with Different Denominators

You can find the sum (or difference) of two fractions with different denominators by first converting them into **equivalent fractions** with the **same denominator**.

This common denominator should be the **least common multiple (LCM)** of the two denominators. The **LCM** of two numbers a and b is the **smallest** possible number that is **divisible** by both a and b .

The **prime factorizations** of a and b can be used to calculate the LCM — the LCM is the product of the highest power of each prime factor that appears in either factorization.

Check it out:

See Topic 2.3.1 for more about **prime factorizations**.

Example 2

Calculate $\frac{2}{3} + \frac{7}{12}$.

Solution

To find the LCM of the denominators, write 3 and 12 as products of prime factors:

$$3 = 3, \quad 12 = 2^2 \times 3$$

So the LCM of 3 and 12 is $3 \times 2^2 = 12$

$$\frac{2}{3} = \frac{8}{12}$$

Convert $\frac{2}{3}$ to an equivalent fraction over 12

$$\frac{8}{12} + \frac{7}{12} = \frac{15}{12}$$

Add fractions with the same denominators

$$\frac{15}{12} = \frac{5 \cdot 3}{4 \cdot 3} = \frac{5}{4}$$

Factor the numerator and denominator and cancel any common factors

So, to add or subtract fractions with different denominators:

1. Find the least common multiple of the denominators.
2. Convert each fraction into an equivalent fraction with the LCM as the denominator.
3. Add or subtract the fractions with the same denominators, then simplify the resultant fraction if possible.

Example 3

Calculate $\frac{5}{8} - \frac{5}{12}$.

Solution

The LCM of 8 and 12 is 24.

$$\frac{5}{8} = \frac{15}{24}, \quad \frac{5}{12} = \frac{10}{24}$$

Convert each fraction into an equivalent fraction with a denominator of 24

$$\frac{15}{24} - \frac{10}{24} = \frac{5}{24}$$

Subtract fractions with the same denominators

✓ Guided Practice

Find the least common multiple of each pair of numbers.

10. 4 and 6

11. 5 and 6

12. 9 and 12

13. 14 and 18

14. 21 and 49

15. 18 and 27

16. 12 and 42

17. 16 and 40

18. 15 and 36

19. 36 and 52

20. 24 and 32

21. 25 and 60

Work out and simplify the following.

22. $\frac{5}{9} + \frac{8}{3}$

23. $\frac{11}{12} + \frac{3}{8}$

24. $\frac{4}{5} + \frac{5}{6}$

25. $\frac{13}{18} + \frac{11}{27}$

26. $\frac{7}{16} + \frac{3}{40}$

27. $\frac{5}{6} - \frac{5}{8}$

28. $\frac{9}{15} - \frac{17}{10}$

29. $\frac{9}{8} - \frac{19}{20}$

30. $\frac{4}{9} - \frac{7}{12}$

✓ Independent Practice

Perform the indicated operations and simplify each expression.

1. $\frac{6}{20} + \frac{8}{20}$

2. $\frac{18}{24} + \frac{7}{24}$

3. $\frac{6}{16} - \frac{4}{16}$

4. $\frac{48}{60} - \frac{18}{60}$

5. $\frac{8}{50} + \frac{3}{10}$

6. $\frac{4}{8} - \frac{5}{12}$

7. $\frac{11}{18} + \frac{4}{12}$

8. $\frac{7}{30} - \frac{3}{24}$

9. $\frac{2}{21} + \frac{7}{11}$

10. Steve and Joe are collecting money to donate to charity.

They set a goal of collecting \$100. Steve has collected \$50, while Joe collected only $\frac{1}{5}$ of the money needed to meet the goal. What fraction of their goal do they still need to collect?

Simplify the following expressions as far as possible.

11. $\frac{3}{7} + \frac{7}{11} - \frac{2}{13}$

12. $\sqrt{1 - \frac{8(x+1)(x-1)}{9x^2 + 6x + 1}}$

Round Up

Fractions will be popping up in all sorts of places throughout Algebra I, so it's a good idea to make sure you really know the rules for them now.

Section 1.4 Mathematical Proofs

California Standards:

24.2: Students identify the hypothesis and conclusion in logical deduction.

25.2: Students judge the validity of an argument according to whether the properties of the real number system and the order of operations have been applied correctly at each step.

What it means for you:

You'll justify each step of mathematical proofs, and you'll learn about the hypothesis and conclusion of "if... then" statements.

Key words:

- justify
- hypothesis
- conclusion
- proof

Check it out:

There are four **properties of equality** — **addition**, **subtraction**, **multiplication**, and **division**. They basically say that you can do the same thing to both sides of an equation. For a formal definition of each, refer to Topic 2.2.1.

*A lot of Algebra I asks you to give **formal proofs** for stuff that you covered in earlier grades. You're sometimes asked to state exactly which **property** you're using for every step of a math problem.*

You Must Justify Each Step of a Mathematical Proof

A mathematical proof is a **logical argument**. When you write a mathematical proof, you have to **justify** each step in a logical way. In Algebra I, you do this using the **axioms** covered earlier in this chapter.

You've seen lots of proofs already in this chapter — although some of them weren't described as proofs at the time. Solving an equation to find the value of a variable is a form of mathematical proof.

Look at the example below. It shows a mathematical proof written in **two columns** — with each step of the logical argument written on the left, and the justification for it written on the right.

Example 1

If $6x + 4 = 22$, what is the value of x ?

Solution

$$6x + 4 = 22$$

$$(6x + 4) - 4 = 22 - 4$$

$$(6x + 4) + (-4) = 22 - 4$$

$$6x + (4 + (-4)) = 22 - 4$$

$$6x + (4 + (-4)) = 18$$

$$6x + 0 = 18$$

$$6x = 18$$

$$\frac{1}{6} \times (6x) = \frac{1}{6} \times 18$$

$$\left(\frac{1}{6} \times 6\right) \cdot x = \frac{1}{6} \times 18$$

$$\left(\frac{1}{6} \times 6\right) \cdot x = \frac{18}{6}$$

$$\left(\frac{1}{6} \times 6\right) \cdot x = 3$$

$$1 \cdot x = 3$$

$$x = 3$$

Given equation

Subtraction property of equality

Definition of subtraction

Associative property of addition

Subtracting

Inverse property of addition

Identity property of addition

Multiplication property of equality

Associative property of multiplication

Definition of division

Dividing

Inverse property of multiplication

Identity property of multiplication

✓ Guided Practice

Complete these statements:

1. A mathematical proof is called a because you have to each step in a logical way using mathematical
2. Mathematical proofs can be written in two columns, with the on the left and the on the right.

Proofs Can Often be Shortened by Combining Steps

Proofs can very often be written in the kind of **two-column format** used in the last example. The next statement in your argument goes on the left, and the justification for it goes on the right. Usually the justification will be something from earlier in this chapter.

However, it's not likely that you'd often need to include every single possible stage in a proof. Usually you'd solve an equation in a few lines, as shown below.

Example 2

If $6x + 4 = 22$, what is the value of x ?

Solution

$$6x + 4 = 22$$

$$6x = 18$$

$$x = 3$$

Check it out:

In Example 2 you're doing several steps at once — but it's the same thing as in Example 1.

Usually it's quicker (and a **much better idea**) to solve an equation the short way, like in Example 2. But you must be **able** to do it the long way if you need to, justifying each step using the real number axioms.

“If..., Then...” Gives a Hypothesis and a Conclusion

Mathematical statements can often be written in the form: “**If..., then...**”

For example, when you solve an equation like the one in Example 2, what you are really saying is: “**If $6x + 4 = 22$, then the value of x is 3.**”

A sentence like this can be broken down into two basic parts — a **hypothesis** and a **conclusion**.

The **hypothesis** is the part of the sentence that follows “**if**” — here, it is $6x + 4 = 22$. The **conclusion** is the part of the sentence that follows “**then**” — here, it is $x = 3$.

IF hypothesis, THEN conclusion.

This doesn't just apply to mathematical statements — it's true for non-mathematical “**If..., then...**” sentences as well. For example:

If an animal is an insect, then it has six legs.
 If you are in California, then you are in the United States.

Now, both the hypothesis and the conclusion can be either true or false. For example, an animal may or may not be an insect, and it may or may not have six legs.

However, the conclusion has to be a **logical consequence** of the hypothesis. Using the example above, this just means that if it *is* an insect, then it *will* have six legs.

Once you've figured out a hypothesis and a conclusion, you can apply the following logical rules:

Check it out:

Watch out — a true conclusion doesn't imply a true hypothesis, and a false hypothesis doesn't imply that the conclusion is false.

If the **hypothesis** is true, then the **conclusion** will also be true.
 If the **conclusion** is false, then the **hypothesis** will also be false.

So if an animal *doesn't* have six legs, then it *isn't* an insect.
 If you *aren't* in the United States, then you're *not* in California.
 And if x is not 3, then $6x + 4 \neq 22$.

Rewrite the following in “If..., then...” format.

1. $4x = 12$ means that $x = 3$.
2. $x + y = 1$ means that $x = 1 - y$.
3. $b + 4 = 17 - y$ means that $b = 13 - y$.

Identify the hypothesis and conclusion in the following statements.

4. If $5y = 30$, then $y = 6$.
5. If $x^2 + y^2 = 16$, then $x^2 = 16 - y^2$.
6. If $d - 12 = 23z$, then $d = 23z + 12$.
7. An animal has four legs if it is a dog.
8. Complete this proof by adding the missing justification steps.

$x - 7 = 17$	Given equation
$(x - 7) + 7 = 17 + 7$
$[x + (-7)] + 7 = 17 + 7$	Definition of subtraction
$[x + (-7)] + 7 = 24$	Adding
$x + [(-7) + 7] = 24$
$x + 0 = 24$	Inverse property of +
$x = 24$

Round Up

The important thing with mathematical proofs is to take each line of the math problem *step by step*. If you're asked to justify your steps, make sure that you state exactly which *property* you're using.

Inductive and Deductive Reasoning

California Standards:

24.1: Students explain the difference between inductive and deductive reasoning and identify and provide examples of each.

24.3: Students use counterexamples to show that an assertion is false and recognize that a single counterexample is sufficient to refute an assertion.

What it means for you:

You'll identify inductive and deductive reasoning in math problems, and you'll find how to use counterexamples to disprove a rule.

Key words:

- inductive reasoning
- deductive reasoning
- counterexample

There are different types of mathematical reasoning.

*Two types mentioned in the California math standards are **inductive reasoning** and **deductive reasoning**.*

Inductive Reasoning Means Finding a General Rule

Inductive reasoning means finding a **general rule** by considering a few specific cases.

For example, look at this sequence of square numbers: **1, 4, 9, 16, 25, 36...**
If you look at the differences between successive terms, you find this:

The difference between the **first** and **second** terms is $4 - 1 = 3$.

The difference between the **second** and **third** terms is $9 - 4 = 5$.

The difference between the **third** and **fourth** terms is $16 - 9 = 7$.

The difference between the **fourth** and **fifth** terms is $25 - 16 = 9$.

If you look at these differences, there's a pattern — each difference is an **odd number**, and each one is **2 greater** than the previous difference.

So using inductive reasoning, you might conclude that:

The difference between **successive square numbers** is always **odd**, and each difference is **2 greater** than the previous one.

Watch out though — this doesn't actually **prove** the rule. This rule does look believable, but to prove it you'd have to use algebra.

Guided Practice

Use inductive reasoning to work out an expression for the n th term (x_n) of these sequences. For example, the formula for the n th term of the sequence 1, 2, 3, 4, ... is $x_n = n$.

1. 2, 3, 4, 5, ...

2. 11, 12, 13, 14, ...

3. 2, 4, 6, 8, ...

4. -1, -2, -3, -4, ...

In exercises 5-6, predict the next number in each pattern.

5. $1 = 1^2$, $1 + 3 = 2^2$, $1 + 3 + 5 = 3^2$, $1 + 3 + 5 + 7 = ?$

6. 1, 1, 2, 3, 5, 8, 13, ...

One Counterexample Proves That a Rule Doesn't Work

If you are testing a rule, you only need to find one **counterexample** (an example that does not work) to prove that the rule is not true.

Once you have found one counterexample, you don't need to look for any more — **one is enough**.

Example 1

Decide whether the following statement is always true:
“ $2^n + 1$ is always a prime number, where n is a natural number.”

Solution

At first, the rule looks believable.

If $n = 1$: $2^n + 1 = 2^1 + 1 = 2 + 1 = 3$.

This is a prime number, so the rule holds for $n = 1$.

If $n = 2$: $2^n + 1 = 2^2 + 1 = 4 + 1 = 5$.

This is a prime number, so the rule holds for $n = 2$.

If $n = 3$: $2^n + 1 = 2^3 + 1 = 8 + 1 = 9$.

This is **not** a prime number, so the rule **doesn't** hold for $n = 3$.

So a counterexample is $n = 3$, and this proves that the rule is **not always true**. Once you've found one counterexample, you don't need to find any more.

✓ Guided Practice

Give a counterexample to disprove each of the following statements.

- All odd numbers are of the form $4n + 1$, where n is a natural number.
- If a number is divisible by both 6 and 3, then it is divisible by 12.
- $|x + 4| \geq 4$
- The difference between any two square numbers is always odd.
- The difference between any two prime numbers is always even.
- The difference between two consecutive cube numbers is always prime.
- All quadrilaterals are squares.
- All angles are right angles.
- All prime numbers are odd.
- A number is always greater than its multiplicative inverse.

Deductive Reasoning Means Applying a General Rule

Deductive reasoning is almost the opposite of inductive reasoning.

In deductive reasoning, you use a general rule to **find out a specific fact**.

Example 2

A number is a multiple of 3 if the sum of its digits is a multiple of 3. Use this information to decide whether 96 is a multiple of 3.

Solution

The sum of the digits is $9 + 6 = 15$, which is divisible by 3.

The statement says that a number is a multiple of 3 if the sum of its digits is a multiple of 3.

Using deductive reasoning, that means that you can say that 96 is divisible by 3.

✓ Guided Practice

Use deductive reasoning to work out the 10th term of these sequences:

17. $x_n = n$

18. $x_n = 6n$

19. $x_n = 2n - 1$

20. $x_n = 3n - 1$

21. $x_n = 20n + 1$

22. $x_n = n(n + 1)$

Use deductive reasoning to reach a conclusion:

23. Ivy is older than Peter. Stephen is younger than Peter.

24. Lily lives in Maryland. Maryland is in the United States.

✓ Independent Practice

Use inductive reasoning in Exercises 1–3.

1. Give the next three numbers of the sequence:

25, 29, 34, 40, ...

2. Write an expression for the n th term (x_n) of the sequence:

24, 72, 216, 648, ...

3. Audrey needs \$650 to buy a digital camera.

Her savings account shows the following balances:

If the pattern continues, at the start of which month will she be able to buy the digital camera?

Date	Balance
Jan 1	\$100.00
Feb 1	\$150.00
March 1	\$250.00
April 1	\$400.00

Use deductive reasoning in exercises 4–5.

4. Find the first five terms of the sequence $x_n = 3n$.

5. Find the first five terms of the sequence $x_n = n(n - 1)$.

Round Up

Inductive reasoning means that you can make a general rule without having to check every single value — so it saves you a lot of work.

Algebraic Statements

California Standards:

25.3: Given a specific algebraic statement involving linear, quadratic, or absolute value expressions or equations or inequalities, students determine whether the statement is true sometimes, always, or never.

What it means for you:

You'll learn how to recognise whether statements are true, and if they're true sometimes or all the time.

Key words:

- hypothesis
- conclusion

Check it out:

See Topic 1.4.1 for more on hypotheses and conclusions. Using the logic from that Topic, if x is not 3 or -3 , then x^2 is not equal to 9.

When you're trying to solve a mathematical problem, sometimes the solution is a *single value*. This Topic is about *other situations*, when things can be a bit more complicated.

Solutions Can Consist of More than One Part

Example 1

Find x given that $x^2 = 9$.

Solution

You can think of the equation as a hypothesis — then you need to find a logical conclusion. One value that satisfies the equation is $x = 3$. However, the statement “If $x^2 = 9$, then $x = 3$ ” is not true — because $x = -3$ also satisfies the equation.

There are two values satisfying the equation, and you need to include **both** of them in your answer.

So the solution is actually: “ $x = 3$ or $x = -3$.”

Or in “If..., then...” form: **If $x^2 = 9$, then $x = 3$ or $x = -3$.**

Some Equations and Inequalities are Always True

If you're given an algebraic statement (such as an **equation** or an **inequality**), there **won't always** be a single value (or even two values) that satisfy the statement.

Example 2

What values of x satisfy $x^2 - 9 = (x + 3)(x - 3)$?

Solution

If you look at the problem above, you might see that $x = 0$ satisfies the equation — because if you put $x = 0$, then both sides equal -9 .

But you might also realize that $x = 3$ satisfies the equation — since if $x = 3$, both sides equal 0.

But that's not all. If $x = -3$, both sides also equal 0.

So $x = -3$ also satisfies the equation.

In fact, the above equation is **always true** — no matter what value you pick for x . So to say “either $x = 0$, $x = 3$, or $x = -3$ ” is **incorrect**. You need to say that it is **always true**.

Sometimes a Statement is Never True

There is another possibility. Look at the following absolute value equation:

Example 3

Find x given that $|x| + 3 = 0$.

Solution

You need to find values for x that satisfy the above equation. However, if you subtract 3 from both sides, you form the equivalent equation:

$$|x| = -3$$

But the absolute value of a number is its distance from 0 on the number line — and a distance cannot be negative. That means that there are no values of x that satisfy the equation.

So the equation is **never true**.

Check it out:

See Topic 1.2.3 for more about absolute value equations.

✓ Guided Practice

1. Find x given that $x^2 = 25$.
2. What values of x satisfy $x^2 - 25 = (x + 5)(x - 5)$?
3. Find x given that $|x| + 5 = 0$
4. Find x given that $x^2 = -16$, $x \in \mathbb{R}$
5. What values of x satisfy $x^2 > 0$?
6. Find x given that $x^2 + 1 = 17$.

More Examples of Mathematical Logic

You can combine the ideas from this section. Look at the following examples.

Example 4

Is the following statement true? If $|a| < |b|$, then $a < b$.

Solution

This is an example where you can find a counterexample to disprove the rule. For example, if $a = -1$ and $b = -2$, then $|a| < |b|$, but $a > b$.

Here, the conclusion isn't a logical consequence of the hypothesis — so you can't apply the logical rules from earlier. That means that the statement is **not true**.

Check it out:

If $|a| < |b|$, then:
 $0 < a < b$, or $b < a < 0$
or $a < 0 < b$, or $b < 0 < a$.

Example 5

If x is a real number, find the possible values for which the following is true: $x^2 + 1 < 2x$

Solution

You can use the inequality as a **hypothesis**. Now you need to find a suitable **conclusion** using a series of logical steps.

$$\begin{aligned} & x^2 + 1 < 2x \\ \Rightarrow & x^2 - 2x + 1 < 0 \\ \Rightarrow & (x - 1)^2 < 0 \end{aligned}$$

So your “If..., then...” statement is: **If $x^2 + 1 < 2x$, then $(x - 1)^2 < 0$.**

Your hypothesis is: $x^2 + 1 < 2x$

Your conclusion is: $(x - 1)^2 < 0$

When you square a real number, you can never get a negative number — so $(x - 1)^2$ cannot be negative.

This means that your **conclusion** is **false**. And using the logic from earlier, this means that your **hypothesis** is also **false**.

So you’ve proved that there is no real number x for which $x^2 + 1 < 2x$. The statement is **never true**.

Example 6

“If a number is odd, its square is odd.”

The number 71774784 is a perfect square. Use the statement above to say whether the square roots of 71774784 are odd.

Solution

Hypothesis: a number is odd

Conclusion: its square is odd

The square of a number is 71774784 — which is an **even** number.

So the conclusion is false.

This means that the hypothesis is also false, and the **square root** of 71774784 is **not odd**.

✓ Independent Practice

1. If x is a real number, find the possible values for which the following is true: $x^2 - 4x + 4 < 0$

2. Is this statement true?

“If $|x| < 3$, then $x < 3$.”

3. Find x given that $x^2 = 64$

4. What values of x satisfy $x^2 - 100 = (x + 10)(x - 10)$?

5. Find x given that $|x| + 5 = 0$.

6. If x is a real number, find the possible values for which the following is true: $x^2 + 9 < 6x$

In exercises 7-11, say whether the algebraic statements are true sometimes, always, or never.

7. $5 + x = 10$

8. $x^2 = -9$

9. $x^2 = 49$

10. $x^2 < 0$

11. $x^2 > x$

12. Is this statement true?

“If $x > 0$, then $x^3 > x^2$ ”

13. Find x given that $|x| + 7 = 0$.

14. What values of x satisfy $x^2 - 25 = (x + 5)(x - 5)$?

15. The area of a rectangle is given by $A = l \times w$ where l and w are the length and width of the rectangle, respectively. Can the area of a rectangle ever be less than zero?

16. Find x given that $x^2 = 81$

17. Is the following statement true?

$$\text{If } x > 0 \text{ then } x^2 > x$$

18. Prove that the following statement is not true:

“the square root of a number is always smaller than the number itself.”

19. Prove that the difference between successive square numbers is always odd, and each difference is 2 greater than the previous difference.

Round Up

*There won't always be a one-part solution to an algebraic statement such as an equation or inequality. If the statement is **always true**, it's no good just giving one value the statement holds for. And if it's **never true**, you have to state that too.*

Chapter 1 Investigation

Counting Collections

Sets and subsets aren't only useful in Math class — they can be used to describe everyday situations.

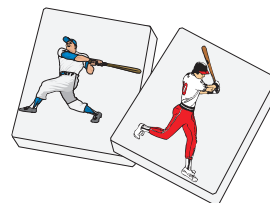
A cereal company is giving away baseball erasers free in their boxes of cereal. There are 7 erasers to collect. All of the children at an elementary school want to collect the whole set. At the moment, they all have different collections and none have more than one of any one eraser.

What is the **maximum number of children** there could be at the school?

Things to think about:

- How many children could there be if there were only **two erasers** to collect?
- How many children could there be if there were **three erasers** to collect?
- How many children could there be if there were **four erasers** to collect?

The collections can overlap — for example, one child could have erasers a , b , and d , while another could have erasers b , c , and g .



Look at your answers — what do you notice?

Extension

- 1) If there were 8 erasers in the set, how many different collections could there be?

What if there were 20 erasers to collect?

Try to find a general rule for the number of different collections for sets with n items.

- 2) A set of trading cards consists of 78 numbered cards.

- How many people could have different collections of cards?
- The cards come in sealed packs that cost \$1.80 per pack.

Each pack contains 8 randomly selected cards. Your friend says that it would only cost \$18 to get the full set. Is your friend right?

- 3) Set G is the set of two-digit prime numbers. How many subsets of G are there?

Round Up

This Investigation shows that you can use sets and subsets to model real-life situations. In fact, you probably divide things into subsets without even realizing it — for example, sorting out your favourite types of candy from a mixed box.

Chapter 2

Single Variable Linear Equations

Section 2.1	Algebra Basics	67
Section 2.2	Manipulating Equations	76
Section 2.3	More Equations	81
Section 2.4	Using Equations	89
Section 2.5	Consecutive Integer Tasks, Time and Rate Tasks	95
Section 2.6	Investment and Mixture Tasks	108
Section 2.7	Work-Related Tasks	121
Section 2.8	Absolute Value Equations	128
Investigation	Wildlife Park Trains	135

Topic 2.1.1

Section 2.1 Simplifying Algebraic Expressions

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.

What it means for you:

You'll combine like terms to simplify expressions.

Key words:

- algebraic expression
- simplify
- like terms

*Unless you're told otherwise, you always need to give algebraic solutions in the **simplest form possible**.*

Algebraic Expressions Contain Variables

Algebraic expressions are made up of **terms**. A term can be a **product of numbers and variables**, like $4x^2$ or $5x$, or just a **number**. For example, the algebraic expression $4x^2 - 5x + 7 - 2x^2 + 2x - 3$ has six terms.

The terms are separated by **plus and minus signs**. Each sign “belongs” to the term that it's in front of.

The diagram shows the algebraic expression $8x^2 + 2xy - 6y + 7y^2 - 2$ with each term circled. An arrow points to the plus sign between $8x^2$ and $2xy$ with the label "Invisible + sign". Another arrow points to the minus sign between $2xy$ and $6y$ with the label "This minus sign belongs to the $6y$ term."

✓ Guided Practice

Write the number of terms in the expressions in Exercises 1–4.

1. $3x^2 + 4x - 2$

2. $8x^4 + 7x^3 + 2x^2 - 8$

3. $3x^2$

4. $8 + 7xy - 2xy^9 + 4x^7 + 3y^2 + 55y^3$

5. Which variable is multiplied by -4 in the algebraic expression $4x^2 - 4y + 8 + 4xy$?

6. Counting from left to right, which term is the fourth term in the algebraic expression $8x^2 + 2xy - 6y + 9xy^3 - 4$?

Like Terms Can Be Combined

Like terms are terms with identical variables that have identical exponents.

The terms $4x^2$ and $-2x^2$ are **like terms** because they have the **same variable**, x , with the **same exponent**, 2. Likewise, $-5x$ and $2x$ are like terms, and 7 and -3 are like terms.

Like terms can be **combined** using the **distributive**, **commutative**, and **associative** properties.

Example 1

Simplify $4x^2 - 2x^2$.

Solution

$$\begin{aligned} &4x^2 - 2x^2 \\ &= (4 - 2)x^2 \\ &= \mathbf{2x^2} \end{aligned}$$

Don't forget:

See Topic 1.3.1 if you're not sure what exponents are.

Combine Like Terms to Simplify Expressions

Don't forget:

See Lesson 1.2.9 for more on number properties.

Check it out:

Subtracting a number is just the same as adding a negative number. So, $4x - 5$ is the same as $4x + (-5)$.

Don't forget:

When there are no more like terms to collect, the expression is simplified as much as possible.

To simplify an algebraic expression, use number properties to first group and then combine like terms.

Example 2

Simplify the following:

a) $(4x - 5) - 2x$ b) $5x^2 - 3x + 7x^2 - 4x + 9$

Solution

a) $(4x - 5) - 2x$
 $= 4x + (-5 - 2x)$
 $= 4x + (-2x - 5)$
 $= (4x + -2x) - 5$
 $= [(4 - 2)x] - 5$
 $= \mathbf{2x - 5}$

Associative property of addition
Commutative property of addition
Associative property of addition
Distributive property

b) $5x^2 - 3x + 7x^2 - 4x + 9$
 $= 5x^2 + 7x^2 - 3x - 4x + 9$
 $= (5x^2 + 7x^2) + (-3x - 4x) + 9$
 $= (5 + 7)x^2 + (-3 - 4)x + 9$
 $= 12x^2 + (-7)x + 9$
 $= \mathbf{12x^2 - 7x + 9}$

✓ Guided Practice

Simplify the following expressions in Exercises 7–14.

7. $7a + 12a - 4$

8. $7a + 3b - 5a$

9. $9x + (20 - 5x)$

10. $5 - 10x - 2x - 7$

11. $7x^2 + 7 + 20x^2 + 3x$

12. $3 - 8x^2 + 4x^2 + 6x^2 - 10$

13. $5a - 4 \times 3a + 7 \times 2 - 3 \times 6$

14. $3 \times 4a - 2 \times 5a^2 + 2 \times 2a^2$

In Exercises 15–18, simplify the expressions and determine the number of terms in each simplified form.

15. $(4a - 9) + (2a - 18)$

16. $15n + 3n + 8 - 2 - 6$

17. $6a^2 + 3a - 9a^2 + 2a + 7 + 6$

18. $6a + 3 \times 7b - 2 \times 5c + 7 - 9 + 2 \times 4c$

✓ Independent Practice

In Exercises 1–5, determine the number of terms in each algebraic expression:

1. $7b + 14a - 4$

2. $2a$

3. $(27x^2 + 4x) - 13$

4. $5 + 10x + 20x^2 + 3a$

5. $2x + 4xy + 4x^2 - (10 + 12y + 19y^2)$

In Exercises 6–9, simplify each algebraic expression:

6. $12m - 7 + 3c - 7m - 8c$

7. $4a + 3b + 11a - 8b$

8. $2 \times 6x - 3 \times 5x - 3x \times 4 + 5 \times 2x + 12x$

9. $5m \times 3 + 2 \times 7m - 4m \times 4 + 7 \times 2m - 17m$

In Exercises 10–13, simplify each algebraic expression:

10. $3x + \frac{1}{2}y - \frac{1}{2}x$

11. $\frac{1}{8}a - \frac{3}{10}b + \frac{8}{10}b - \frac{6}{12}a$

12. $(17.8n + 13.08q) - 3.8q - 9.9n$

13. $0.4x^2 + \frac{3}{8}x - \frac{1}{2}x^2 - 0.14 - \frac{5}{8}$

14. Which expression below simplifies to $2x + 1$?

i. $x + 4x + 4 - 3 - 3x + x$

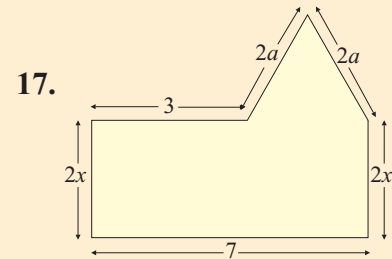
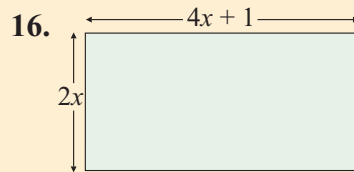
ii. $7 + 5x - 6 + 5 - 4x + x - 5$

15. Which expression below simplifies to $3x$?

i. $5 + 4x^2 - 3x - 2 + 4x^2 - 8x^2 - 3$

ii. $-2x^2 + 6 + 4x^2 - 3 + 3x - 2x^2 - 3$

In Exercises 16–17, find a simplified expression for the perimeter of the figure.



18. Juan bought 3 baseball cards for b dollars each and 2 baseball cards for c dollars each. He has bought 4 comic books for \$5.00 each. Write and simplify an algebraic expression showing the total money Juan spent on baseball cards and comic books.

19. Three friends Tom, Leo, and Maria have several pieces of candy to eat. Tom has $(2x + 4)$ pieces of candy, Leo has $(8 - 2x)$ pieces of candy, and Maria has 8 pieces of candy. Write and simplify an algebraic expression showing the total number of pieces of candy the three friends have to eat.

Round Up

You've combined like terms before, in earlier grades — so this Topic should feel like good practice. It's always important to give your final answers to algebraic problems in the simplest form.

Getting Rid of Grouping Symbols

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.

What it means for you:

You'll use the distributive property to simplify expressions.

Key words:

- distributive property
- commutative property

Don't forget:

After getting rid of the grouping symbols, collect the like terms together.

You already saw the *distributive property* in Topic 1.2.7. In this Topic you'll simplify expressions by using the distributive property to *get rid of grouping symbols*.

The Distributive Property Removes Grouping Symbols

The expression $5(3x + 2) + 2(2x - 1)$ can be **simplified** — both parts have an “ x ” term and a constant term.

To simplify an expression like this, you first need to get rid of the **grouping symbols**. The way to do this is to use the **distributive property** of multiplication over addition: $a(b + c) = ab + ac$.

Example 1

Simplify $5(3x + 2) + 2(2x - 1)$.

Solution

$$\begin{aligned} &5(3x + 2) + 2(2x - 1) \\ &= 15x + 10 + 4x - 2 \\ &= 15x + 4x + 10 - 2 \\ &= 19x + 8 \end{aligned}$$

Distributive property
Commutative property of addition

✓ Guided Practice

In Exercises 1–7, simplify the following expressions:

1. $2(4x + 5) + 8$
2. $12(5a - 8) + 4x + 3$
3. $6(2j + 3c) + 8(5c + 4z)$
4. $10(x + 2) + 7(3 - 4x)$
5. $6(a - b) + 4(2b - 3)$
6. $5(3x + 4) + 3(4x + 10) + 2(8x + 9)$
7. $8(2n - 3) + 9(4n - 5) + 4(3n + 7)$

Take Care when Multiplying by a Negative Number

If a number outside a grouping symbol is **negative**, like in $-7(2x + 1)$, you have to remember to use the **multiplicative property of -1** .

This means that the signs of the terms **within** the grouping symbols will **change**: “+” signs will change to “-” signs and vice versa.

Example 2

Simplify the following:

a) $-7(2x + 1)$ b) $-6(-x - 3)$ c) $-3(5x - 4)$

Solution

a) The $+2x$ and $+1$ become negative.

$$-7(2x + 1) = -14x - 7$$

b) The two negative terms inside the grouping symbols are multiplied by the negative term outside. They both become positive.

$$-6(-x - 3) = 6x + 18$$

c) $-3(5x - 4) = -15x + 12$

Example 3

Simplify the expression $4(2x - 1) - 5(x - 2)$. Show your steps.

Solution

$$\begin{aligned} &4(2x - 1) - 5(x - 2) \\ &= 8x - 4 - 5x + 10 \\ &= 8x - 5x - 4 + 10 \\ &= 3x + 6 \end{aligned}$$

Given expression

Distributive property

Commutative property of addition

Check it out

$3x + 6$ has no more like terms, so it can't be simplified any further.

✓ Guided Practice

In Exercises 8–13, simplify each algebraic expression:

8. $-2(5a - 3c)$

9. $-8(3c - 2)$

10. $-2(-3x - 4) + 4(6 - 2x)$

11. $7(2a + 9) - 4(a + 11)$

12. $-8(2y + 4) - 5(y + 4)$

13. $-2\left(\frac{1}{2}n + 2\right) - 3\left(\frac{1}{3}n - 4\right)$

14. Simplify $12(2n - 7) - 9(3 - 4n) + 6(4x - 9)$.

15. Simplify $5(x - 2) - 7(-4x + 3) - 3(-2x)$.

✓ Independent Practice

In Exercises 1–6, simplify the algebraic expressions:

1. $-4(a + 2b)$

2. $3(2n + 4) + n(-4)$

3. $5(3b - 2q) - (3q + 4)$

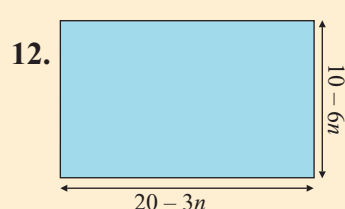
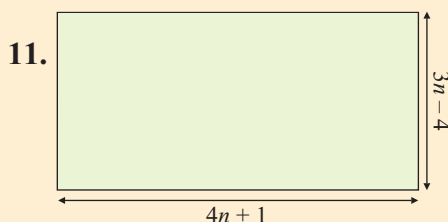
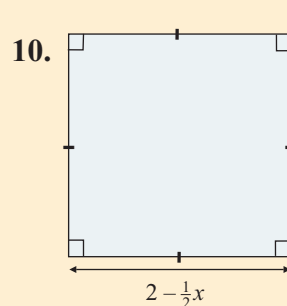
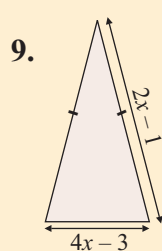
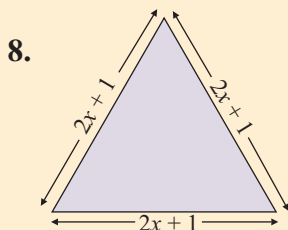
4. $-9(2 + 3b) + 3(3 - 2b)$

5. $-17(3a - 5b) - 4(x + 3)$

6. $-2(10n - 4x) + 2(n + 6x) - 3(7x - 2n)$

7. Simplify $-7(3p - 9q) - 4(25q - 3r) + 12(5p + 8)$.

In Exercises 8–12, find and simplify an expression for the perimeter of the shape shown.



In Exercises 13–15, simplify the algebraic expressions:

13. $\frac{3}{8}(3x + 4) - \frac{1}{2}(4x - 8)$

14. $-0.1(25n + 46) - 0.8(16n - 5) + 0.4(3 - 4n)$

15. $-\frac{1}{3}(5 - 9x) - \frac{1}{8}(2x - 4) - 0.2(-7x - 10)$

16. Mia has $4(x + 1)$ dolls, where x is Mia's age. Madeline has $5(2x + 3)$ dolls. Write and simplify an algebraic expression showing the total number of dolls in Mia and Madeline's collection.

17. Ruby, Sara, and Keisha are counting stamps. If x represents Ruby's age, she has $4(x - 4)$ stamps, Sara has $2(8 - x)$ stamps, and Keisha has $8(7 + 2x)$ stamps. Write and simplify an algebraic expression showing the total number of stamps owned by the three friends.

Round Up

The *distributive property* is really useful — it's always good to get rid of confusing grouping symbols whenever you can. The main thing you need to watch out for is if you're multiplying the contents of parentheses by a *negative number* — it will *change the sign* of everything in the parentheses.

More Simplifying and Checking Answers

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.

What it means for you:

You'll use the rules of exponents to simplify expressions, and you'll show that proposed solutions to equations are correct.

Key words:

- exponents
- commutative property
- equation

Don't forget:

After getting rid of the grouping symbols, collect any like terms together.

Check it out:

$2xy$ and yx are like terms.

This Topic gives another method of simplifying expressions — using the rules of exponents.

Use Rules of Exponents to Multiply Variables

To simplify expressions like $4x(x^2 - 2x + 1)$, you need to apply the distributive property as well as rules of exponents, such as $x^a \times x^b = x^{a+b}$.

Example 1

Simplify $-2x(x^2 - 2y + 1) - x(-4xy + y)$.

Solution

$$\begin{aligned} & -2x(x^2 - 2y + 1) - x(-4xy + y) \\ &= -2x^3 + 4xy - 2x + 4x^2y - xy \\ &= -2x^3 + 3xy - 2x + 4x^2y \end{aligned}$$

**Get rid of grouping symbols first
Then collect like terms**

If you have two or more **different** variables multiplied together, it doesn't matter what order they're in. For example, xy is the same as yx , and ab is the same as ba . This is because of the **commutative property of multiplication**.

Example 2

Simplify $2x(y + 1) - y(x + 3)$.

Solution

$$\begin{aligned} & 2x(y + 1) - y(x + 3) \\ &= 2xy + 2x - yx - 3y \\ &= 2xy - yx + 2x - 3y \\ &= xy + 2x - 3y \end{aligned}$$

✓ Guided Practice

In Exercises 1–9, simplify the algebraic expressions:

- $2x(3x - 4)$
- $-4x(x - 4)$
- $-2y(3yx + 2)$
- $2x(x + 5y) + 3y(y + 3)$
- $2y(2x + 2) - 4(2x + 2)$
- $6y(yx - 4) + 5(yx - 4)$
- $7n(3a + b) - 4a(7n + 2b)$
- $2x(2x^2 - x) + x(2x - 8) + 3x(x - 4)$
- $2x(k - 9) - k(x - 7) + xk(4 - 3x)$

Two Equal Expressions Can Be Written as an Equation

Don't forget:

See Topic 2.2.1 for the difference between expressions and equations.

For any equation: **Left-hand side = Right-hand side**

When you think you know what numbers the variables in an equation represent, you should always put them into the equation to check that both sides of the equation are equal.

Replace the variables with the **numbers** and make sure that the left side of the equation is **equal to** the right side of the equation.

You do exactly the same thing if a question asks you to **prove** or **show** that an equation is true.

Example 3

Show that $x = -1$ is a solution of $4(x - 2) + 2 = 4x - 2(2 - x)$.

Solution

$$4(x - 2) + 2 = 4x - 2(2 - x)$$

You're told that $x = -1$ is a solution, so replace each x with -1 .

$$\Rightarrow 4(-1 - 2) + 2 = 4(-1) - 2[2 - (-1)]$$

$$\Rightarrow 4(-3) + 2 = 4(-1) - 2(3)$$

$$\Rightarrow -12 + 2 = -4 - 6$$

$$\Rightarrow -10 = -10$$

This shows that $4(x - 2) + 2 = 4x - 2(2 - x)$ is a true statement for $x = -1$.

Check it out:

Continue simplifying until you end up with the same number on each side of the equation.

Guided Practice

10. Given $x = -1$, show that $2(5x + 3) - 3(3x + 2) = 4x + 3$.

11. Given $b = -1$, show that $\frac{4(2b-3)}{3} - \frac{3(2b-6)}{4} - \frac{5b}{9} = -\frac{1}{9}$

12. Given $x = -\frac{5}{3}$, show that $4(2x - 1) - 5(x - 2) = 1$.

13. Show whether $x = -2$ is or is not a solution of $-6x - 15 = -17 - 9x$.

14. If $b = -10$, show that $\frac{b-2}{6} - \frac{b-3}{5} = \frac{3}{5}$.

15. Show whether $x = -\frac{2}{3}$ is or is not a solution of $-6x - 15 = -17 - 9x$.

16. Verify that $x = 0$ is a solution of $\frac{3}{8}(3x + 4) - \frac{1}{2}(4x - 8) = 5.5$.

17. Verify that $x = -1$ is a solution of $(4x - 9)^2 = 169$.

18. Verify that $x = -7$ is a solution of $0.2(\frac{4}{9}x + 3) - 3(\frac{2}{9}x - \frac{1}{3}) = 5\frac{29}{45}$.

Equations may Contain Two or More Variables

Sometimes your equation will have **two or more different variables**.

Example 4

Given that $a = 3$, $b = -2$, and $c = 0$, show that $-2a(4 + b) = b(5 - c) - 2$.

Solution

Again, just substitute the numbers for the variables and simplify until you show that both sides of the equation are equal.

$$-2a(4 + b) = b(5 - c) - 2$$

$$\Rightarrow -2(3)(4 - 2) = -2(5 - 0) - 2$$

$$\Rightarrow -6(2) = -2(5) - 2$$

$$\Rightarrow -12 = -10 - 2$$

$$\Rightarrow -12 = -12$$

$\therefore -2a(4 + b) = b(5 - c) - 2$ for the given values of a , b , and c .

✓ Guided Practice

In Exercises 19–24, find the value of each algebraic expression when the given substitutions are made:

19. $2x(x + 5y) - 3y(y + 3)$ if $x = 2$, $y = 4$

20. $6y(yx - 4) - 5(yx - 4)$ if $x = 1$, $y = -1$

21. $7n(30 + b) - 4a(7n + 2b)$ if $a = 0$, $b = 3$, $n = \frac{1}{7}$

22. $-2y(3yx + 2)$ if $x = 4$, $y = -8$

23. $-4a(b - 4)$ if $a = 4$, $b = 0.2$

24. $2x(k - a) - k(x - a) + xk(a - 3x)$ if $a = -4.2$, $x = 0.1$, $k = \frac{1}{3}$

✓ Independent Practice

In Exercises 1–3, simplify the algebraic expressions:

1. $-7a(8bc - 3a)$

2. $6n(yn + 7) - 7n(n - yn)$

3. $2ab(c - d) + 4cb(c - d) - 3ac(2b + d)$

4. Find the value of $2a(4a - 3) + (8 - a)$, if $a = 3$.

5. Find the value of $2ab(c - d) + 4cb(c - d) - 3ac(2b + d)$, if $a = 8$, $b = 7$, $c = 6$, $d = 0$.

6. The formula $P = 6.5h + 0.10x$ is used to find the weekly pay of a salesperson at a local electronic store, where P is the pay in dollars, h is the number of hours worked, and x is the total value of merchandise sold (in dollars) by the salesperson. If the salesperson worked 40 hours and sold \$4,250 worth of merchandise, how much pay did she earn?

Round Up

The material on equations in this Topic leads neatly on to the next Section.

The rest of the Sections in this Chapter are all about *forming and manipulating equations*.

Section 2.2 Properties of Equality

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.

What it means for you:

You'll solve one-step equations using properties of equality.

Key words:

- expression
- equation
- variable
- solve
- isolate
- equality

Now it's time to use the material on expressions you learned in Section 2.1. An **equation** contains *two expressions*, with an *equals sign* in the middle to show that they're *equal*.

In this Topic you'll solve equations that involve addition, subtraction, multiplication, and division.

An Equation Shows That Two Expressions are Equal

An **equation** is a way of stating that two expressions have the same value. This equation contains only numbers — there are no unknowns:

The expression on the left-hand side... $\rightarrow 24 - 9 = 15 \leftarrow$...has the same value as the expression on the right-hand side

Some equations contain unknown quantities, or **variables**.

The left-hand side... $\rightarrow 2x - 3 = 5 \leftarrow$...equals the right-hand side

The value of x that satisfies the equation is called the **solution** (or **root**) of the equation.

Addition and Subtraction in Equations

Addition Property of Equality

For any real numbers a , b , and c , if $a = b$, then $a + c = b + c$.

Subtraction Property of Equality

For any real numbers a , b , and c , if $a = b$, then $a - c = b - c$.

These properties mean that adding or subtracting the same number on both sides of an equation will give you an **equivalent equation**. This may allow you to **isolate the variable** on one side of the equals sign.

Finding the possible values of the variables in an equation is called **solving the equation**.

Example 1

Solve $x + 9 = 16$.

Solution

You want x on its own, but here x has 9 added to it. So subtract 9 from both sides to get x on its own.

$$(x + 9) - 9 = 16 - 9$$

$$x + (9 - 9) = 16 - 9$$

$$x + 0 = 16 - 9$$

$$x = 16 - 9$$

$$x = 7$$

In Example 1, $x = 7$ is the root of the equation. If x takes the value 7, then the equation is **satisfied**.

If x takes **any other value**, then the equation is **not satisfied**.

For example, if $x = 6$, then the left-hand side has the value $6 + 9 = 15$, which does not equal the right-hand side, 16.

When you're actually solving equations, you won't need to go through all the stages each time — but it's really important that you understand the theory of the properties of equality.

- If you have a “+ 9” that you don't want, you can get rid of it by just **subtracting 9** from both sides.
- If you have a “- 9” that you want to get rid of, you can just **add 9** to both sides.

In other words, you just need to use the **inverse operations**.

Example 2

Solve $x + 10 = 12$.

Solution

$$x + 10 = 12$$

$$x = 12 - 10 \quad \text{Subtract 10 from both sides}$$

$$x = 2$$

Example 3

Solve $x - 7 = 8$.

Solution

$$x - 7 = 8$$

$$x = 8 + 7 \quad \text{Add 7 to both sides}$$

$$x = 15$$

Guided Practice

In Exercises 1–8, solve the equation for the unknown variable.

1. $x + 7 = 15$

2. $x + 2 = -8$

3. $\frac{4}{3} + x = \frac{2}{3}$

4. $x - (-9) = -17$

5. $-9 + x = 10$

6. $x - 0.9 = 3.7$

7. $\frac{4}{9} = x - \frac{1}{3}$

8. $-0.5 = x - 0.125$

Multiplication and Division in Equations

Multiplication Property of Equality

For any real numbers a , b , and c ,
if $a = b$, then $a \times c = b \times c$.

Division Property of Equality

For any real numbers a , b , and c ,
such that $c \neq 0$, if $a = b$, then $\frac{a}{c} = \frac{b}{c}$.

These properties mean that multiplying or dividing by the same number on both sides of an equation will give you an **equivalent equation**.

That can help you to **isolate the variable** and solve the equation.

Multiply or Divide to Get the Variable on Its Own

As with addition and subtraction, you can get the variable on its own by simply **performing the inverse operation**.

- If you have “ $\times 3$ ” on one side of the equation, you can get rid of that value by **dividing** both sides by 3.
- If you have a “ $\div 3$ ” that you want to get rid of, you can just **multiply** both sides by 3.

Once again, you just need to use the **inverse operations**.

Example 4Solve $2x = 18$.**Solution**You want x on its own... but here you've got $2x$.

$$\frac{2x}{2} = \frac{18}{2}$$

Divide both sides by 2 to get x on its own

$$\frac{2}{2}x = 9$$

$$1x = 9$$

$$x = 9$$

Example 5Solve $\frac{m}{3} = 7$.**Solution**

$$\frac{m}{3} \times 3 = 7 \times 3$$

Multiply both sides by 3 to get m on its own

$$m = 21$$

Some equations are a bit more complicated. Take them **step by step**.**Example 6**Solve $4x - 2(2x - 1) = 2 - x + 3(x - 4)$.**Solution**

$$4x - 2(2x - 1) = 2 - x + 3(x - 4)$$

$$4x - 4x + 2 = 2 - x + 3x - 12$$

$$2 = -10 + 2x$$

$$2x = 12$$

$$x = 6$$

Clear out any grouping symbols
Then combine like terms**Guided Practice**

In Exercises 9–16, solve each equation for the unknown variable.

9. $4x = 144$

10. $-7x = -7$

11. $\frac{x}{4} = -7$

12. $\frac{3}{2x} = 9$

13. $-3x + 4 = 19$

14. $4 - 2x = 18$

15. $3x - 2(x - 1) = 2x - 3(x - 4)$

16. $3x - 4(x - 1) = 2(x + 9) - 5x$

✓ Independent Practice

Solve each of these equations:

1. $-2(3x - 5) + 3(x - 1) = -5$

2. $4(2a + 1) - 5(a - 2) = 8$

3. $5(2x - 1) - 4(x - 2) = -15$

4. $2(5m + 7) - 3(3m + 2) = 4m$

5. $4(5x + 2) - 5(3x + 1) = 2(x - 1)$

6. $b - \{3 - [b - (2 - b) + 4]\} = -2(-b - 3)$

7. $4[3x - 2(3x - 1) + 3(2x - 1)] = 2[-2x + 3(x - 1)] - (5x - 1)$

8. $30 - 3(m + 7) = -3(2m + 27)$

9. $8x - 3(2x - 3) = -4(2 - x) + 3(x - 4) - 1$

10. $-5x - [4 - (3 - x)] = -(4x + 6)$

In Exercises 11–17, solve the equations and check your solutions. You don't need to show all your steps.

11. $4t = 60$

12. $x + 21 = 19$

13. $\frac{p}{8} = 1$

14. $7 - y = -11$

15. $\frac{y}{7} = 4$

16. $40 - x = 6$

17. $\frac{s}{-4} = -4$

18. Solve $-(3m - 8) = 12 - m$

19. Denzel takes a two-part math test. In the first part he gets 49 points and in the second part he gets $\frac{4}{9}$ of the x points. If his overall grade for the test was 65, find the value of x .

20. Latoya takes 3 science tests. She scores 24%, 43%, and $x\%$ in the tests. Write an expression for her average percentage over the 3 tests. Her average percentage is 52. Calculate the value of x .

Solve the following equations. Show all your steps and justify them by citing the relevant properties.

21. $x + 8 = 13$

22. $11 + y = 15$

23. $y - 7 = 19$

24. $\frac{x}{3} = 4$

25. $4m = 16$

Round Up

Solving an equation means *isolating the variable*. Anything you don't want on one side of the equation can be "taken over to the other side" by using the *inverse operation*. You're always aiming for an expression of the form: " $x = \dots$ " (or " $y = \dots$ " etc.).

Section 2.3 Removing Fractions

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.

What it means for you:

You'll use the least common multiple to remove fractional coefficients from equations.

Key words:

- coefficient
- least common multiple

Expressions often seem more complicated if they contain fractions.

Getting rid of fractions isn't too difficult — you just need to use the least common multiple (LCM) again.

Removing Fractions Makes Solving Equations Easier

To solve equations that contain **fractional coefficients**, you can get rid of all the fractional coefficients by **multiplying** both sides of the equation by any **common multiple** of the denominators of the fractions. You don't have to remove fractions, but it can make solving the equation a lot easier.

The most efficient thing to multiply by is the **least common multiple** (LCM).

If you needed to solve $\frac{1}{4}x - 1 = \frac{1}{6}x$, you first need to multiply by the least common multiple of 4 and 6. To find the LCM, list the prime factors:

$$4 = 2 \times 2 \qquad 6 = 2 \times 3$$

Now write each prime factor the greatest number of times it appears in any of the factorizations.

The prime factor 2 occurs twice in the factorization of 4, so count two of them. The prime factor 3 occurs only once in the factorization of 6, so just count one of them. Then multiply the factors together to get the LCM:

$$2 \times 2 \times 3 = 12$$

So the LCM of 4 and 6 is **12**.

Example 1

Solve the equation $\frac{1}{4}x - 1 = \frac{1}{6}x$.

Solution

The LCM of the denominators is 12 — so multiply both sides of the equation by 12:

$$\frac{12}{1} \times \frac{1}{4}x - \frac{12}{1} \times 1 = \frac{12}{1} \times \frac{1}{6}x$$

$$3x - 12 = 2x$$

$$3x - 2x = 12$$

$$x = 12$$

✓ Guided Practice

In Exercises 1–2, find the least common multiple of the denominators:

1. $\frac{1}{9}x + 13 - \frac{1}{4}x = 4 - \frac{1}{6}x$

2. $\frac{1}{2}x - 4 + \frac{1}{5}x = \frac{1}{3}x + 1$

In Exercises 3–6, solve the equations for the unknown variable.

3. $\frac{1}{2}x - 1 = x$

4. $\frac{1}{10}x - 3 = \frac{1}{4}x$

5. $\frac{1}{10}x - 1 = 8 - \frac{4}{5}x$

6. $4 - \frac{1}{3}x = \frac{1}{5}x + 8$

You Can Work Out the LCM for Two or More Fractions

Example 2

Solve and check the root of $\frac{2}{3}x - \frac{5}{6}x - 3 = \frac{1}{2}x - 5$.

Solution

LCM of 3, 6, and 2 is 6.

$$\frac{6}{1} \times \frac{2}{3}x - \frac{6}{1} \times \frac{5}{6}x - \frac{6}{1} \times 3 = \frac{6}{1} \times \frac{1}{2}x - \frac{6}{1} \times 5$$

$$2 \times 2x - 1 \times 5x - 6 \times 3 = 3 \times 1x - 6 \times 5$$

$$4x - 5x - 18 = 3x - 30$$

$$-x - 18 = 3x - 30$$

$$-x - 3x = -30 + 18$$

$$-4x = -12$$

$$\frac{-4x}{-4} = \frac{-12}{-4}$$

$$x = 3$$

Checking the solution:

$$\frac{2}{3}x - \frac{5}{6}x - 3 = \frac{1}{2}x - 5$$

$$\frac{2}{3} \times 3 - \frac{5}{6} \times 3 - 3 = \frac{1}{2} \times 3 - 5$$

$$\frac{6}{3} - \frac{15}{6} - 3 = \frac{3}{2} - 5$$

$$2 - \frac{5}{2} - 3 = \frac{3}{2} - 5$$

$$-1 - \frac{5}{2} = \frac{3}{2} - 5$$

$$\frac{-2 - 5}{2} = \frac{3 - 10}{2}$$

$$-\frac{7}{2} = -\frac{7}{2}$$

Check it out:

Since the left side of the equation is equal to the right side of the equation, $x = 3$ is a correct solution of the equation.

✓ Guided Practice

In Exercises 7–14, solve your answer for the unknown variable:

$$7. \frac{2}{3}x - 5 + \frac{1}{4}x = 2 - \frac{1}{4}x$$

$$8. \frac{2}{3}x - \frac{5}{2}x = \frac{5}{2}x + 22 - \frac{2}{3}x$$

$$9. \frac{1}{8}n - \frac{1}{8} + \frac{1}{4}n = \frac{1}{2} - \frac{1}{4}n - \frac{1}{8}n$$

$$10. 8 = -\frac{3}{5}(2x - 1) + 5$$

$$11. \frac{1}{2}x + \frac{1}{5}x + \frac{1}{8}x = 17 - \frac{1}{2}x - \frac{3}{8}x$$

$$12. \frac{1}{2} + \frac{7}{10}x - \frac{1}{5}x + 17 = \frac{10}{20} + \frac{3}{2}x$$

$$13. \frac{1}{4}(a - 16) + \frac{3}{5} = 7 - \frac{1}{15}(a + 4)$$

$$14. \frac{1}{2}m + \frac{5}{2}(m - 1) - \frac{7}{2} = \frac{3}{2} - \frac{3}{8}(1 + m)$$

✓ Independent Practice

In Exercises 1–2, find the least common multiple of the denominators:

$$1. \frac{1}{7}x - 3 = \frac{1}{2}x - \frac{1}{3}x + 6$$

$$2. \frac{1}{9}x + \frac{1}{7}x - 3 = 2 + \frac{1}{3}x$$

In Exercises 3–8, solve the equation for the unknown variable:

$$3. \frac{1}{2}x + 4 = \frac{1}{3}x$$

$$4. \frac{1}{4}x + \frac{1}{6} = \frac{1}{9}x + 6$$

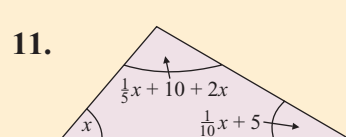
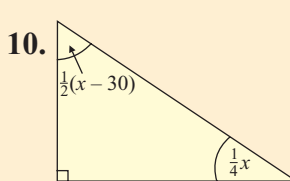
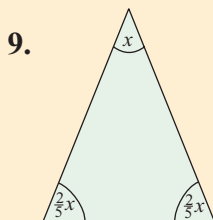
$$5. \frac{1}{5}x - \frac{1}{10}x = \frac{1}{3}x + 7$$

$$6. \frac{1}{7}x - 3 = \frac{1}{21}x - \frac{1}{3}x + 6$$

$$7. \frac{1}{9}x + \frac{1}{7}x - 3 = 2 + \frac{1}{3}x$$

$$8. \frac{2}{5}(m - 2) - \frac{1}{5}m = -\frac{1}{5}$$

The sum of the measures of the angles of a triangle is 180° . In Exercises 9–11, find the value of x .



12. Qiaofang bought a new car for \$27,000. If the value of the car depreciates as $27,000 - \frac{7100}{3}n$, where n is the number of years since purchase, when will it be worth \$12,800?

13. Mary's weekly allowance increases every year as $\frac{2}{5}x + \$2$, where x is Mary's age. How old is Mary if she gets \$5.20 allowance?

Round Up

If you multiply one side of an equation by a value, then you need to multiply the other side by the same thing. Make sure that the least common multiple is a multiple of the denominator of every fraction in the equation — otherwise you'll still end up with annoying fractions in the equation.

Fractional Coefficients in Algebraic Expressions

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.

What it means for you:

You'll use the least common multiple to remove fractional coefficients from equations.

Key words:

- coefficient
- least common multiple

Don't forget:

See Lesson 2.3.1 for more on the least common multiple.

Check it out:

Use parentheses to group each numerator before you multiply by the LCM.

In the last Topic you dealt with *equations with fractional coefficients* — but sometimes the coefficients apply to more complicated values.

Eliminate Fractional Coefficients First

Some equations contain fractional coefficients — and you can't isolate the variables until you **sort out the fractions**.

$(x-2)$ is all divided by 5. You need to deal with the fraction before you can get x on its own. $\Rightarrow \frac{x-2}{5} - \frac{x-3}{6} = \frac{3}{10}$ \Leftarrow The denominators are all different, so multiply the equation by the LCM of all three denominators.

If there are fractional coefficients in the equation, **multiply** both sides of the equation by the **least common multiple** of the denominators of the fractions to remove the fractional coefficients.

You need to keep each numerator (the “ $x-2$,” “ $x-3$,” and “3”) as a **group**. Then you can apply the **distributive property** to eliminate the grouping symbols.

Example 1

Solve $\frac{x-2}{5} - \frac{x-3}{6} = \frac{3}{10}$.

Solution

There are 3 different denominators, so you need the LCM. List the prime factors, and use them to work out the LCM.

$$5 = 5 \quad 6 = 2 \times 3 \quad 10 = 2 \times 5$$

So LCM = $2 \times 3 \times 5 = 30$

Multiply both sides of the equation by 30 to clear the fractional coefficients.

$$\frac{30}{1} \times \frac{(x-2)}{5} - \frac{30}{1} \times \frac{(x-3)}{6} = \frac{30}{1} \times \frac{3}{10}$$

$$6(x-2) - 5(x-3) = 3 \times 3$$

$$6x - 12 - 5x + 15 = 9$$

$$6x - 5x - 12 + 15 = 9$$

$$x + 3 = 9$$

$$x = 6$$

Example 2

Solve $\frac{11}{12} - \frac{x-1}{6} = \frac{x+1}{4}$.

Solution

$$\frac{11}{12} - \frac{(x-1)}{6} = \frac{(x+1)}{4}$$

$$\frac{12}{1} \times \frac{11}{12} - \frac{12}{1} \times \frac{(x-1)}{6} = \frac{12}{1} \times \frac{(x+1)}{4}$$

$$11 - 2(x-1) = 3(x+1)$$

$$11 - 2x + 2 = 3x + 3$$

$$-5x = -10$$

$$x = 2$$

Check it out

$$12 = 2 \times 2 \times 3$$

$$6 = 2 \times 3$$

$$4 = 2 \times 2$$

$$\text{So LCM} = 2 \times 2 \times 3 = 12$$

Independent Practice

Solve each of the following equations:

1. $-\frac{3}{4}v + 7 = -\frac{2}{3}v + 6$

2. $-\frac{3}{8}x - \frac{1}{2} = \frac{3}{4}$

3. $\frac{2}{5}y + \frac{1}{10} = \frac{3}{2}y$

4. $\frac{m-2}{5} - \frac{m-3}{6} = \frac{3}{10}$

5. $\frac{5(2y-1)}{6} + \frac{11}{12} = \frac{3(4y+1)}{8}$

6. $\frac{6x+5}{3} = \frac{4x+1}{2} + \frac{5x}{6}$

7. $\frac{4(3x+2)}{5} - \frac{3(4x-1)}{10} = -\frac{7}{10}x$

8. $\frac{4(2y-3)}{3} = \frac{3(2y-5)}{4} + \frac{5}{6}y$

9. $\frac{1}{4}(2c-1) - \frac{1}{6}(c+2) = \frac{c}{3} - \frac{1}{12}(2c-3)$

10. $\frac{b+2}{3} - \frac{3-b}{5} = \frac{5b+2}{10}$

11. $\frac{l-7}{4} - \frac{l+5}{5} = -\frac{3}{4}$

12. $\frac{2x+7}{3} - 4 + \frac{3x-8}{7} = 9 - \frac{2(x-5)}{21} + 2$

Round Up

This Topic is really important, because you can't simplify an expression fully until you've got rid of any fractional coefficients. Luckily, you've got the LCM to help you.

Eliminating Decimal Coefficients

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.

What it means for you:

You'll remove decimal coefficients from equations by multiplying by a suitable power of 10.

Key words:

- coefficient
- least common multiple

If the coefficients are *decimal*, you still just need to multiply the equation by a suitable number — in fact, this time it's slightly easier.

Get Rid of Decimals by Multiplying by a Power of 10

Solving an equation that contains decimals can be made easier if it is first converted to an **equivalent equation** with **integer coefficients**. The method is the same as with fractions — you just **multiply** both sides of the equation by a suitable number.

The idea is to multiply both sides of the equation by a large enough **power of 10** to convert all decimals to integer coefficients.

For example, 0.35 means “35 hundredths,” so you can write it $\frac{35}{100}$. So multiplying by 100 gives you the integer 35.

Example 1

Solve the equation $0.35x - 12 = -0.15x$.

Solution

Multiply both sides of the equation by 100:

$$\begin{aligned} 100(0.35x - 12) &= 100(-0.15x) \\ 100 \times 0.35x - 100 \times 12 &= 100 \times -0.15x \\ 35x - 1200 &= -15x \\ 50x &= 1200 \\ \mathbf{x} &= \mathbf{24} \end{aligned}$$

Example 2

Solve $0.75x - 0.65(13 - x) = 8.35$.

Solution

Multiply both sides of the equation by 100:

$$\begin{aligned} 100[0.75x - 0.65(13 - x)] &= 100(8.35) \\ 100(0.75x) - 100(0.65)(13 - x) &= 100(8.35) \\ 75x - 65(13 - x) &= 835 \\ 75x - 845 + 65x &= 835 \\ 75x + 65x &= 835 + 845 \\ 140x &= 1680 \\ \mathbf{x} &= \mathbf{12} \end{aligned}$$

✓ Guided Practice

In Exercises 1–10, solve the equation for the unknown variable:

- $0.04x - 0.12 = 0.01x$
- $-0.01x + 0.3 = 0.03x + 0.18$
- $0.06x - 0.09 = 0.05x + 0.12 - 0.15$
- $0.11 - 0.03x - 0.03 = 0.02x - 0.02$
- $0.16m + 2 = 0.03(5m + 2)$
- $0.01a = 0.45 - (0.04a + 0.15)$
- $0.25(x + 4) + 0.10(x - 2) = 3.60$
- $0.10(x + 100) = 0.08(x - 6)$
- $0.25(x - 50) = 0.2(x - 10) + 5$
- $0.25(x + 8) + 0.10(8 - x) + 0.05x = 3.60$

Sometimes You Need to Multiply by Higher Numbers

In Examples 1 and 2 you needed to multiply both sides of each equation by 100. You **can't** always just multiply by 100, though. If any of the decimals have **more than 2 decimal places**, you'll need a **higher** power of 10.

Example 3

Solve $0.015x = 0.2 - 0.025x$.

Solution

Multiply both sides by 1000 this time.

$$1000(0.015x) = 1000(0.2 - 0.025x)$$

$$15x = 200 - 25x$$

$$40x = 200$$

$$x = 5$$

If your longest decimal has **3 decimal places**, multiply by $10^3 = 1000$.
If the longest decimal has **4 decimal places**, multiply by 10^4 , and so on.

✓ Guided Practice

In Exercises 11–20, solve each equation for the unknown variable.

- $0.3(2x + 7) = 1.8$
- $2.8 = 0.2(4x + 2)$
- $0.016 = 0.002(a - 1) - 0.001a$
- $0.1x + 0.1(3x - 8) = -230$
- $0.001x + 0.05(2x - 3) = 35.2$
- $0.012p - 0.065p - 0.7 = 0.5$
- $0.003(x + 7) = 0.01x$
- $0.006(x + 3) = 0.002(x + 31)$
- $0.072 - 0.006j = 0.032 + 0.08j$
- $0.003(2x - 0.3) + 0.001x = 0.1979$

Independent Practice

Solve each of the equations in Exercises 1–17:

- $0.11x - 2.3 + 0.4(2x - 1) = 0.25(3x + 8)$
- $0.5x - 1 = 0.7x + 0.2$
- $0.3v + 4.2v - 11 = 1.5v + 4$
- $1.8(y - 1) = 3.1y + 2.1$
- $0.25(3x - 2) + 0.10(4x + 1) + 0.75(x - 1) = 4.55$
- $0.125(3b - 8) - 0.25(b + 5) - 0.2 = 0.2(2b + 7)$
- $0.4(x + 7) - 0.15(2x - 5) = 0.7(3x - 1) - 0.75(4x - 3)$
- $0.21(x - 1) - 0.25(2x + 1) = 0.5(2x + 1) - 0.6(4x - 15) + 0.03$
- $0.9(2v - 5) + 0.20(-3v - 1) = 0.22(4v - 3)$
- $2.46 - 0.52(x - 10) = 0.35(4x + 8) - 2.82$
- $-0.20x - 1.10(3x - 11) = 1.25(2x - 5) - 6.25$
- $16 + 0.50y = 0.60(y + 20)$
- $20 + 1.20m = 1.10(m + 25)$
- $0.03x + 0.30(900 - x) = 72$
- $0.06y + 0.3y - 0.1 = 0.26$
- $0.04k + 0.06(40,000 - k) = 2100$
- $-0.02[0.4 - 0.1(2 + 3x)] = 0.004x + 0.005$
- The sides of an equilateral triangle measure $0.2(10x + 90)$ units each. If the perimeter of the triangle is 612.6 units, find x .
- The area of a rectangle is 34 units². If the width is 0.25 units and the length is $(x + 10)$ units, find the value of x .
- The cost per minute to make a call is \$0.05. If Meimei talks for $(x + 20)$ minutes and the call costs \$4.85, what is the value of x ?
- A cell phone plan charges \$25 per month plus \$0.10 per minute. If your monthly bill is \$39.80, write and solve an equation to find out the number of minutes on your bill.
- A moving van rents for \$40 a day plus \$0.08 a mile. Ed's bill is \$58.24 and he had the van for one day. Write and solve an equation to find out how many miles he drove.
- Eylora has x quarters with a value of $\$0.25x$. Emily has dimes that value $0.10(x + 8)$. If they have a total of \$5.00 in coins, how many coins does Eylora have?
- Michael, William, and Daniel are playing a game. Michael has x points, William has $0.01(x + 18,000)$ points, and Daniel has $0.02(x - 800)$ points. If together they have earned 11,288 points, how many points does Michael have?

Round Up

Whether you're dealing with *fractional* or *decimal coefficients*, the method's essentially the same — you *multiply* everything by a number that will make the algebra easier and mistakes less likely. Then you can start *isolating the variable*.

Applications of Linear Equations

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.

5.0: Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

What it means for you:

You'll set up and solve equations that model real-life situations.

Key words:

- application
- linear equation

Check it out:

Look out for words that give math clues — for example, “twice” means two times and the word “is” means equals.

“Applications” are just “real-life” tasks.

In this Topic, linear equations start to become really useful.

Applications of Equations are “Real-Life” Tasks

Applications questions are **word problems** that require you to set up and solve an equation.

- First decide how you will label the variables...
- ...then write the task out as an equation...
- ...making sure you include all the information given...
- ...then you can solve your equation.

Example 1

The sum of twice a number c and 7 is 21.

Set up and solve an equation to find c .

Solution

You're given the label “ c ” in the question — so just write out the equation.

$$2c + 7 = 21$$

$$2c = 14$$

$$c = 7$$

✓ Guided Practice

1. Twice a number c plus 17 is 31. Find the value of c .
2. Three times a number k minus 8 is 43. Find the number k .
3. The sum of four times a number m and 17 is the same as 7 less than six times the number m . Find the number m .
4. Seven minus five times the number x is equal to the sum of four times the number x and 25. Find the number x .

You Won't Always Be Given the Variables

Sometimes you'll have to work out for yourself what the **variables** are, and decide on suitable labels for them.

Example 2

Juanita's age is 15 more than four times Vanessa's age. The sum of their ages is 45. Set up and solve an equation to find their ages.

Solution

This time you have to decide for yourself how to label each term.

Let v = Vanessa's age

$4v + 15$ = Juanita's age

$$v + (4v + 15) = 45$$

The sum of their ages is 45

$$v + 4v + 15 = 45$$

$$5v + 15 = 45$$

$$5v = 30$$

$$v = 6$$

Plug in the value for v to get Juanita's age:

$$4v + 15$$

$$= 4 \times 6 + 15$$

$$= 24 + 15$$

$$= 39$$

So Vanessa is 6 years old and Juanita is 39 years old.

Check it out:

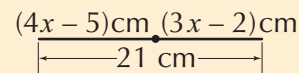
" $4v + 15$ " is just the phrase
"15 more than four times v "
written in math-speak.

Guided Practice

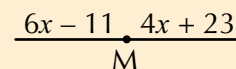
- The length of a rectangular garden is 3 meters more than seven times its width. Find the length and width of the garden if the perimeter of the garden is 70 meters.
- A rectangle is 4 meters longer than it is wide. The perimeter is 44 meters. What are the dimensions and area of the rectangle?
- Abraham's age is 4 less than half of Dominique's age. Dominique's age is 6 more than three times Juan's age. The sum of their ages is 104. Find the age of each person.
- The sides of an isosceles triangle are each 2 inches longer than the base. If the perimeter of the triangle is 97 inches, what are the lengths of the base and sides of the triangle?
- The sum of two consecutive integers is 117. What are the integers?

✓ Independent Practice

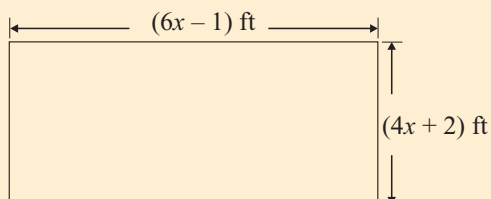
1. Find the value of x , if the line segment shown on the right is 21 cm long. Also, find the length of each part of the line segment.



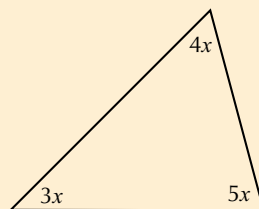
2. Point M is the midpoint of the line segment shown. Find the value of x and the length of the entire line segment.



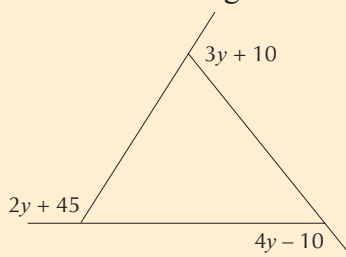
3. The perimeter of the rectangular plot shown below is 142 feet. Find the dimensions of the plot.



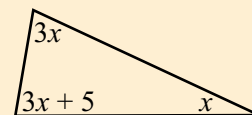
4. The sum of the interior angles of a triangle is 180° . Find the size of each angle in the triangle shown below.



5. The sum of one exterior angle at each vertex of any convex polygon is 360° . Find the size of each exterior angle shown around the triangle below.



6. The interior angles of a triangle sum to 180° . Find the size of each angle in the triangle sketched on the right.



7. A rectangular garden has a length that is five meters less than three times its width. If the length is reduced by three meters and the width is reduced by one meter, the perimeter will be 62 meters. Find the dimensions of the garden.

Round Up

The thing to do with any *word problem* is to write out a *math equation* that describes the same situation. Then you can use all the techniques you've learned to *solve* the equation.

Coin Tasks

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.

5.0: Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

What it means for you:

You'll set up and solve equations that model real-life situations involving money.

Key words:

- coin task

Check it out:

Convert everything in the question to the same units — in this example it's all been converted into cents.

*Lots of math problems involve objects of value such as **coins** and **stamps**. They're often grouped together under the title "coin tasks."*

Solving coin tasks uses all the same methods that you used in the last Topic.

Coin Tasks Involve Linear Equations About Money

If you have a collection of coins, there are two quantities you can use to describe it — **how many** coins there are, and **how much they are worth**. In coin tasks you'll have to use both quantities.

Example 1

Jazelle has a coin collection worth \$3.50. She only has nickels, dimes, and quarters. If she has four more dimes than quarters and twice as many nickels as she has dimes, how many coins of each kind does she have in her collection?

Solution

The first thing to do is to write expressions for how many of each type of coin she has.

- Call the number of quarters q .
- Then the number of dimes is $q + 4$.
- And the number of nickels is $2(q + 4) = 2q + 8$.

Then write expressions for the total value (in cents) of each type of coin.

- There are q quarters, so the total value of the quarters is $25q$.
- There are $q + 4$ dimes, so the total value of the dimes is $10(q + 4) = 10q + 40$.
- There are $2q + 8$ nickels, so the total value of the nickels is $5(2q + 8) = 10q + 40$.

Write an equation to represent the fact that the collection is worth \$3.50.
 $25q + (10q + 40) + (10q + 40) = 100(3.50)$

Solving for q gives:

$$45q + 80 = 350$$

$$45q = 270$$

$$q = 6$$

So Jazelle has:

$$q = 6 \text{ quarters,}$$

$$q + 4 = 6 + 4 = 10 \text{ dimes,}$$

$$2(q + 4) = 2(6 + 4) = 20 \text{ nickels.}$$

Example 2

Tickets to a puppet show sell at \$2.50 for children and \$4.50 for adults. There are five times as many children at a performance as there are adults and the show raises \$3009. How many adult and children's tickets were sold for the show?

Solution

Let x = adult tickets sold.

Then $5x$ = children's tickets sold.

The value is the number of tickets multiplied by the cost of each ticket.

So $4.50x$ = amount raised from adults (in dollars)

$5x(2.50)$ = amount raised from children

Write an equation showing that the sum of these amounts is \$3009:

$$4.50x + 5x(2.50) = 3009$$

$$17x = 3009$$

$$x = 177$$

So **177 adult tickets** and $5 \times 177 =$ **885 children's tickets** were sold.

 **Independent Practice**

1. The total cost of buying some music CDs and having them shipped to Charles was \$211.50. If the CDs cost \$11.50 each and the shipping for the box of CDs was \$4.50, how many CDs did Charles receive?
2. Jerome bought 12 CDs. Some of the CDs cost \$7.50 each and the rest cost \$6.50 each. How many CDs were bought at each price if Jerome spent a total of \$82?
3. Rajan's coin collection is valued at \$22.70. He has one fewer half-dollar coin than twice the number of dimes and four more quarters than three times the number of dimes. How many dimes, quarters, and half-dollars does Rajan have, assuming he has no other coins?
4. Liza has 60 coins in her collection. The coin collection consists of nickels, dimes, and quarters. She has five fewer quarters than nickels and ten more dimes than quarters. How many coins of each kind does Liza have?
5. A school cafeteria cashier has collected \$243 in one-dollar, five-dollar, and ten-dollar bills. The number of one-dollar bills is eight more than 20 times the number of ten-dollar bills. The cashier also has seven more than twice the number of ten-dollar bills in five-dollar bills. How many bills of each value does the cashier have?

6. Dan's algebra class is planning a summer afternoon get-together. Dan is supposed to bring some melons at \$1.25 each, juice boxes at \$0.50 each, and granola bars for \$0.75 each. If he buys nine more than ten times the number of melons in juice boxes and seven more than five times the number of melons in granola bars, how many items of each kind did he buy with \$29.75?
7. Martha bought some baseball uniforms for \$313 and had them shipped to her. If the baseball uniforms cost \$23.75 each and the shipping was \$4.25 for the whole order, how many baseball uniforms did Martha buy?
8. Fifteen children's books cost \$51.25. Some were priced at \$2.25 each, and the rest of the books were sold at \$4.75 each. How many books were purchased at each price?
9. Dwight purchased various stamps for \$16.35. He purchased 12 more 25¢ stamps than 35¢ stamps. The number of 30¢ stamps was four times the number of 35¢ stamps. Finally, he bought five 15¢ stamps. How many of each kind of stamp did Dwight buy?
10. A waiter has collected 150 coins from the tips he receives from his customers. The coins consist of nickels, dimes, and quarters. He has five more than twice the nickels in dimes and five more than four times the nickels in quarters.
- i) How many coins of each kind does the waiter have?
ii) How much money does the waiter have?
11. Jessica has two more nickels than dimes, and three more quarters than nickels, but no other coins. If she has a total of \$5.35, how many coins of each kind does she have?
12. A grocer's deposit box contains 150 coins worth \$12.50. They are all nickels and dimes. Find the number of each coin in the box.
13. A store sells nineteen different video games. Several games are priced at \$19.99, while half that number are priced at \$39.99, and 4 are priced at \$49.99. How many games are priced at \$19.99?
14. Mark bought packets of popcorn, drinks, and bags of nuts for him and some friends at the movie theater. Everyone got one of each. Drinks cost \$4.50 each, packets of popcorn cost \$3.75 each, and bags of nuts cost \$2.00 each. If he spent \$41, how many friends did Mark have with him?
15. John has quarters, nickels, and dimes. He has 4 more nickels than quarters and twice as many dimes and nickels. If he has \$6.00, how many quarters does he have?

Round Up

To answer these questions, you need to use (i) the number of items, and (ii) their value. Then, when you have set up your equation and solved it, be sure to give your final answer in the form asked for in the problem.

Section 2.5 Consecutive Integer Tasks

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.
5.0: Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

What it means for you:

You'll solve word problems that refer to patterns of integers.

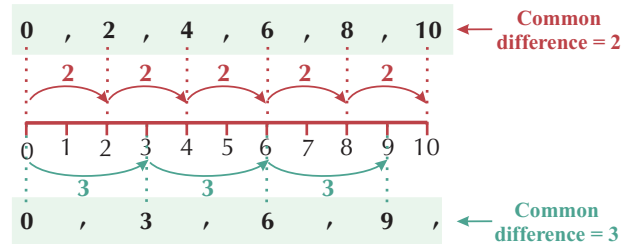
Key words:

- consecutive
- integer
- common difference

When you're solving word problems, the most important thing to do is to *write down what you know*. Then create an equation that represents the relationship between the known and unknown quantities. Finally, you have to *solve that equation*.

Sequences with a Common Difference

A sequence of integers with a **common difference** is a set of integers that increase by a fixed amount as you move from term to term.



Consecutive **EVEN** integers (for example, 2, 4, and 6) form a sequence with common difference **2**.

Consecutive **ODD** integers (for example, 1, 3, and 5) also form a sequence with common difference **2**.

So if you are given an even integer (or an odd integer) and you are asked to find the next even (or odd) integer, just **add two**.

Example 1

Find the three consecutive even integers whose sum is 48.

Solution

Call the first (smallest) even integer x . Then you can write down an expression for the other even integers in terms of x .

$$\text{1st even integer} = x$$

$$\text{2nd even integer} = x + 2$$

$$\text{3rd even integer} = x + 4$$

But the sum of the three even integers is 48, so $x + (x + 2) + (x + 4) = 48$.

Now you can rearrange and simplify this formula, and then solve for x .

$$x + (x + 2) + (x + 4) = 48$$

$$(x + x + x) + (2 + 4) = 48$$

$$3x + 6 = 48$$

$$3x = 42$$

$$x = 14$$

Check it out:

Add 2 to x to find the second even integer, and another 2 to find the third.

Example 1 continued

So the smallest of the three even integers is 14. Now just add 2 to find the second even integer, and add another 2 to find the third.

So the three consecutive even integers are:

$$x = 14, x + 2 = 16, \text{ and } x + 4 = 18.$$

Write Expressions for the Unknowns First

The most important thing with any word problem is to first write it out again in math-speak.

Example 2

Find a sequence of four integers with a common difference of 4 whose sum is 92.

Solution

Call the first (smallest) integer v , for example. Then you can write the other three in terms of v .

$$\text{1st integer} = v$$

$$\text{2nd integer} = v + 4$$

$$\text{3rd integer} = v + 8$$

$$\text{4th integer} = v + 12$$

The sum of the four integers is 92, so:

$$v + (v + 4) + (v + 8) + (v + 12) = 92$$

Now you can solve for v :

$$(v + v + v + v) + (4 + 8 + 12) = 92$$

$$4v + 24 = 92$$

$$4v = 68$$

$$v = 17$$

So the smallest of the integers is 17.

Since they differ by 4, the others must be 21, 25, and 29.

Once you've completed the problem, do a quick answer check — add the integers together and see if you get what you want:

$$\text{1st integer} = v = 17$$

$$\text{2nd integer} = v + 4 = 21$$

$$\text{3rd integer} = v + 8 = 25$$

$$\text{4th integer} = v + 12 = \frac{29}{92} \checkmark$$

Example 3

Find a sequence of four integers with a common difference of 2 whose sum is 944. Decide whether these integers are even or odd.

Solution

Call the first integer x , then write expressions for the others:

$$\text{1st integer} = x$$

$$\text{2nd integer} = x + 2$$

$$\text{3rd integer} = x + 4$$

$$\text{4th integer} = x + 6$$

The sum of the four integers is 944, so:

$$x + (x + 2) + (x + 4) + (x + 6) = 944$$

Now solve for x :

$$4x + 12 = 944$$

$$4x = 932$$

$$x = 233$$

So the smallest of the integers is **233**, which means the others must be **235**, **237**, and **239**. These are consecutive **odd** integers.

Now do a quick answer check — add the integers together and see if you get what you want:

$$\begin{array}{rcl} \text{1st integer} & = x & = 233 \\ \text{2nd integer} & = x + 2 & = 235 \\ \text{3rd integer} & = x + 4 & = 237 \\ \text{4th integer} & = x + 6 & = \underline{239} \\ & & \hline & & 944 \quad \checkmark \end{array}$$

Check it out:

The difference between the integers is 2, whether they are odd or even.

✓ Guided Practice

Find the unknown numbers in each of the following cases.

1. The sum of two consecutive integers is 103.
2. The sum of three consecutive integers is -138 .
3. The sum of two consecutive even integers is 194.
4. The sum of three consecutive odd integers is -105 .
5. The sum of a number and its double is 117.
6. Two integers have a difference of three and their sum is 115.
7. Three integers have a common difference of 3 and their sum is -78 .
8. Four rational numbers have a common difference of 5 and a sum of 0.

Independent Practice

1. Three consecutive integers have a sum of 90. Find the numbers.
2. Find the four consecutive integers whose sum is 318.
3. Find three consecutive integers such that the difference between three times the largest and two times the smallest integer is 30.
4. Find three consecutive integers such that the sum of the first two integers is equal to three times the highest integer.
5. Two numbers have a sum of 65. Four times the smaller number is equal to 10 more than the larger number. Find the numbers.
6. Four consecutive even integers have a sum of 140.
What are the integers?
7. Find three consecutive even integers such that six more than three times the smallest integer is 54.
8. Find three consecutive odd integers whose sum is 273.
9. Find four consecutive odd integers such that 12 more than four times the smallest integer is 144.
10. Find three consecutive even integers such that six more than twice the first number is 94.
11. Find three consecutive even integers such that the product of 16 and the third integer is the same as the product of 20 and the second integer.
12. A 36-foot pole is cut into two parts such that the longer part is 11 feet longer than 4 times the shorter part. How long is each piece of the pole?
13. Find three consecutive odd integers such that four times the largest is one more than nine times the smallest integer.
14. Ten thousand people attended a three-day outdoor music festival. If there were 800 more girls than boys, and 1999 fewer adults than boys, how many people of each group attended the festival?

Round Up

*Consecutive integer tasks are a strange application of math equations — but they appear a lot in Algebra I. Always make sure you've answered the question — you've always got to remember that your solution isn't complete until you've stated **what the integers actually are.***

Age-Related Tasks

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.
5.0: Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

What it means for you:

You'll solve equation problems that refer to people's ages.

Key words:

- age task

Age tasks relate ages of different people at various time periods — in the past, the present, or the future.

Age-Related Tasks

Your age at any point in your life can always be written as your **current age plus or minus** a certain number of years.

If your current age is x years...	x
...then 5 years ago, your age was 5 fewer than x ...	$x - 5$
...and in 5 years' time, your age will be 5 more than x .	$x + 5$
More generally, your age c years ago was:	$x - c$
And your age in c years will be:	$x + c$

In much the same way, anybody's age can always be related to someone else's by adding or subtracting a certain number of years.

Write Expressions for the Unknown Quantities

Solving an age task is pretty similar to solving a consecutive integer task. You need to write down expressions for the unknown quantities in terms of one variable (like x). Then you can use the information in the question to **write an equation** that you can go on to **solve**.

Example 1

Charles is 7 years older than Jorge. In 20 years' time, the sum of their ages will be 81 years. How old is each one now?

Solution

The first thing to do is write expressions relating all the ages to one another.

Present

You're not told Jorge's age, so call it x .

Charles is currently 7 years older — that is, 7 more than x .

$$\text{Jorge's age} = x$$

$$\text{Charles's age} = x + 7$$

Future (in 20 years)

Jorge will be 20 years older than at present.

Charles will be 20 years older than at present.

Example 1 continued

$$\text{Jorge's future age} = x + 20$$

$$\text{Charles's future age} = (x + 7) + 20 = x + 27$$

Now use the information from the question to combine these expressions into an equation. The sum of their ages in 20 years will be 81, so:

$$(x + 20) + (x + 27) = 81$$

Now solve your equation for x :

$$(x + 20) + (x + 27) = 81$$

$$2x + 47 = 81$$

$$2x = 34$$

$$x = 17$$

So Jorge is currently **17 years old**.

That means that Charles is $17 + 7 =$ **24 years old**.

Example 2

Juanita is twice as old as Vanessa. If 5 years were subtracted from Vanessa's age and 2 years added to Juanita's age, then Juanita's age would be five times Vanessa's. How old are the girls now?

Solution

As before, start by writing down mathematical expressions for the ages mentioned in the question.

Present

You're not given Vanessa's age, so call it v .

Juanita is twice as old — that is, twice v .

$$\text{Vanessa's age} = v$$

$$\text{Juanita's age} = 2v$$

“Adjusted” ages

$$\text{“If 5 years is subtracted from Vanessa's age...”} = v - 5$$

$$\text{“If 2 years is added to Juanita's age...”} = 2v + 2$$

Now you can write an equation. Remember that Juanita's “adjusted” age is five times as big as Vanessa's.

$$2v + 2 = 5(v - 5)$$

Now solve for v to find Vanessa's current age:

$$2v + 2 = 5(v - 5)$$

$$2v + 2 = 5v - 25$$

$$-3v = -27$$

$$v = 9$$

So Vanessa is **9 years old**.

And since Juanita is twice as old, Juanita is $2 \times 9 =$ **18 years old**.

Check it out:

Watch out — you haven't answered the question unless you say what $v = 9$ means for the people's ages.

Example 3

A father is three times as old as his daughter. In 15 years' time, the father will be twice as old as his daughter. What are their current ages?

Solution**Present**

You are not given the daughter's age, so call it x .

The father is three times as old — three times x .

$$\text{Daughter's age} = x$$

$$\text{Father's age} = 3x$$

Future (in 15 years)

The daughter and father will both be 15 years older.

$$\text{Daughter's future age} = x + 15$$

$$\text{Father's future age} = 3x + 15$$

In 15 years, the father's age will be twice as great as his daughter's.

Write this as an equation, and solve:

$$3x + 15 = 2(x + 15)$$

$$3x + 15 = 2x + 30$$

$$x = 15$$

So the daughter is **15 years old**, and the father is $3 \times 15 =$ **45 years old**.

Don't forget:

Remember to say what your value of x means in the context of the question.

✓ Guided Practice

- Eylora is 4 times as old as Leo. If the sum of their ages is 5, how old is Eylora?
- Clarence is 8 years older than Maria. In 24 years, the sum of their ages will be 100. How old is Clarence?
- Tyler, Nick, and Sid are brothers. The sum of their ages is 54. The oldest brother, Nick, is 2 years older than Sid, and Sid is 2 years older than Tyler. How old is Sid?
- A father is 9 times older than his daughter and 2 years older than his wife. If the sum of their ages is 74, how old is the father?
- Ruby and Emily are twins. Rebecca is 6 more than 2 times Ruby and Emily's age. Altogether the sum of their three ages is 50. How old are Emily and Ruby?
- Santos is 30 years older than his daughter Julia. If their ages are increased by 10% and added together the sum is 77. How old is Santos?
- James is 4 years less than 7 times the age of his daughter, who is 4 more than half of her brother's age. The sum of their ages is 38. How old is James?

Independent Practice

1. Keisha, Juan, and Jose are friends who are all different ages. There is a 2 year difference in age between the oldest and youngest. Juan is not as old as Jose, but he is older than Keisha. If the sum of their ages is 36, how old is the oldest child?
2. Sonita's father is three times as old as she is now. In ten years, her father will be twice as old as Sonita will be then. How old are Sonita and her father now?
3. Kadeeja is four times as old as her niece. In three years, Kadeeja will be three times as old as her niece. How old is each of them now?
4. Sally is twice as old as Daniel. Ten years ago, the sum of their ages was 70 years. How old is each one of them now?
5. Andy is four times as old as Alejandro. Five years ago, Andy was nine times as old as Alejandro. How old is each one now?
6. Chris is 40 years younger than his uncle. In ten years' time the sum of their ages will be 80 years. How old are they now?
7. Jorge is three times Martha's age. If 30 years is added to Martha's age and 30 years is subtracted from Jorge's age, their ages will be equal. How old is each person now?
8. Mia's age in 20 years will be the same as Simon's age is now. Ten years from now, Simon's age will be twice Mia's age. How old is each one now?
9. Paula is three times as old as Duncan. If four is subtracted from Duncan's age and six is added to Paula's age, Paula will then be four times as old as Duncan. How old are they now?
10. If you decrease Marvin's age by 25%, you will find his age 4 years ago. How old is Marvin now?
11. If you increase Qin's age by 75% then you will find his age 6 years from now. How old will Qin be in 6 years?
12. Jaya is 10 years younger than Sid. If you increase both of their ages by 20%, the difference between their ages is 18 less than Jaya's current age. How old is Sid?

Round Up

Age tasks are just another example of real-life equations. As always, you have to set up an equation from the information you're given, then solve the equation. Your answer is only complete when you include the actual ages of the people involved.

Rate, Time, and Distance Tasks

California Standards:

15.0: Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

What it means for you:

You'll learn the formulas for rate, distance, and time, and use them to set up equations to solve real-life problems involving rates.

Key words:

- rate
- speed
- distance
- time

*The greater an object's **speed**, the greater the distance it **travels** in a given amount of time. **Rate, time, and distance tasks** are a little different to the ones you've seen in this Section so far because they have particular **formulas** that you need to learn.*

Speed, Time, and Distance are Related by a Formula

The quantities of **distance**, **time**, and **speed** are related by a formula.

Speed is the **distance traveled per unit of time**

— for instance, the distance traveled in one second, one hour, etc.

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

The units of speed depend on the units used for the distance and time.

For example, if the distance is in **miles** and the time is in **hours**, then the speed will be in **miles per hour**.

You Can Rearrange the Speed, Distance, Time Formula

The above formula can be rearranged to give these important formulas for distance and time:

$$\text{distance} = \text{speed} \times \text{time}$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

✓ Guided Practice

Use the speed, distance, and time formulas to work out the following:

1. Find the average speed if distance = 116 miles and time = 2 hours.
2. Find the average speed if distance = 349 km and time = 5 h.
3. Find the distance if speed = 75 mph and time = 2.5 h.
4. A car completes a 125 mile journey traveling at an average speed of 50 miles per hour. Work out the time taken.
5. A train travels a distance of 412.5 km at an average speed of 120 km per hour. How long did the journey take?
6. A train travels at an average speed of 140 mph. If it takes 4 hours to reach its destination, how far did it travel?
7. A long-distance runner completes a half marathon (13.1 miles) in a time of 1 hour 45 minutes. Find the runner's average speed.

Write Down What You Know, Then Solve an Equation

Motion tasks normally involve **two objects** with **different speeds**.

Example 1

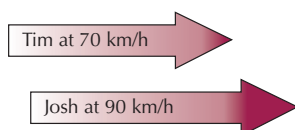
Tim drives along a road at 70 km/h. Josh leaves from the same point an hour later and follows exactly the same route.

If Josh drives at 90 km/h, how long will it take for Josh to catch up with Tim?

Solution

As always, first write down what you know — and use sketches, arrows, and tables if they help you visualize what is happening.

Drawing arrows helps remind you which way each person is traveling:



Tim left first...

...but Josh is traveling faster (and in the same direction), so he will catch up.

Suppose Josh catches Tim x hours after Josh left.

At that time, Tim will have been traveling for $(x + 1)$ hours.

	Time (in hours)	Speed (in km/h)	Distance (in km)
Josh	x	90	$90x$
Tim	$x + 1$	70	$70(x + 1)$

= speed \times time
Time is in hours and speed is in kilometers per hour — so all distances will be in kilometers.

Josh catches Tim when they have both traveled the same distance.

So you need to solve:

$$90x = 70(x + 1)$$

$$90x = 70x + 70$$

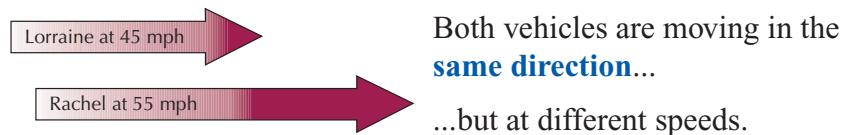
$$20x = 70$$

$$x = 3.5$$

So it will take Josh **3.5 hours** to catch up with Tim.

Example 2

Lorraine is driving to a theme park at 45 miles per hour. Twenty minutes after Lorraine leaves, Rachel sets off along the same freeway. If Rachel is traveling at 55 miles per hour, how long does it take her to catch up with Lorraine?

Solution

Lorraine left first, but she's traveling slower. Rachel left afterwards but is traveling faster — so at some point Rachel will catch up.

If you call Rachel's travel time x (hours), then Lorraine's travel time will be $\left(x + \frac{1}{3}\right)$ (since Lorraine left 20 minutes $(= \frac{1}{3}$ of an hour) earlier).

	Time (in hours)	Speed (in mph)	Distance (in miles)
Rachel	x	55	$55x$
Lorraine	$x + \frac{1}{3}$	45	$45\left(x + \frac{1}{3}\right)$

= speed \times time

When Rachel catches up, Rachel and Lorraine have traveled equal distances, so $55x = 45\left(x + \frac{1}{3}\right)$. So solve this equation to find x :

$$55x = 45x + \frac{45}{3}$$

$$10x = 15$$

$$x = \frac{3}{2}$$

$$x = 1\frac{1}{2}$$

So, Rachel catches up with Lorraine $1\frac{1}{2}$ hours after Rachel left.

✓ Guided Practice

8. Felipe sets off from Phoenix for Los Angeles, driving at 60 mph. Thirty minutes later an emergency vehicle takes off after Felipe, on the same route, at an average speed of 80 mph. How long will it take the emergency vehicle to overtake Felipe?
9. Lavasha and Keisha travel separately to their grandmother's house, 114 miles away. Keisha leaves from the same place 30 minutes after Lavasha. If Lavasha is traveling at 40 mph and Keisha travels at 55 mph, how long will it take Keisha to catch up with Lavasha?
10. An express train leaves Boston for Washington D.C., traveling at 110 mph. Two hours later, a plane leaves Boston for Washington D.C., traveling at a speed of 550 mph. If the plane flies above the train route, how long will it take the plane to pass the express train?

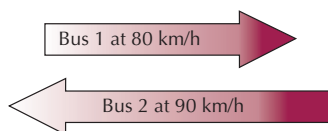
Watch Out for Things Moving in Opposite Directions

When you're solving motion questions, you always need to work out which **direction** each of the objects is moving in. If things are traveling in opposite directions, you have to think carefully about the equation.

Example 3

Bus 1 leaves Bulawayo at 8 a.m. at a speed of 80 km/h. It is bound for Harare, 680 km away. Bus 2 leaves the Harare depot at 8 a.m., heading for Bulawayo along the same highway at a speed of 90 km/h. Calculate at what time the buses pass each other.

Solution



This time the buses are traveling in **opposite directions**.

	Time (in hours)	Speed (in km/h)	Distance (in km)
Bus 1	x	80	$80x$
Bus 2	x	90	$90x$

The initial distance between the two buses is 680 km, and they pass when the distance between them is zero. This means they must pass when the distances they have traveled sum to 680 km.

So solve $80x + 90x = 680$. This gives $170x = 680$, or $x = 4$.

Now you have to say what this value of x means.

Since they pass 4 hours after their departure time of 8 a.m., they must pass at **noon**.

Check it out:

Careful — the question asked for the time the buses pass, not for how long they will have been traveling.

Guided Practice

11. An express train leaves Chicago for Atlanta at 100 mph. At the same time a freight train leaves Atlanta for Chicago at 80 mph. If the distance between the two cities is 720 miles, how long will it take the trains to pass each other?
12. Ernesto, a resident of Los Angeles, sets off to visit his uncle in San Francisco. At the same time, Chang, a resident of San Francisco, sets off to visit a friend in Los Angeles. The road between San Francisco and Los Angeles is 390 miles long. If Ernesto drives at 60 mph while Chang drives at 70 mph, how long will it take for the drivers to pass each other?
13. Mr. and Mrs. Ding leave their home traveling in opposite directions on a straight road. Mrs. Ding drives 10 mph faster than Mr. Ding. After 2 hours, they are 200 miles apart. Find the rates that Mr. and Mrs. Ding are traveling.

Independent Practice

1. A plane leaves Miami for Seattle at 9:00 a.m., traveling at 450 mph. At the same time another plane leaves Seattle for Miami, flying at 550 mph. At what time will the planes pass if the distance between Miami and Seattle is 3300 miles?
2. A jet leaves the airport traveling at a speed of 560 km/h. Another jet, leaving the same airport and traveling in the same direction, leaves 45 minutes later traveling at 750 km/h. About how long will it take for the second jet to overtake the first jet?
3. Two planes depart Los Angeles at the same time. One plane flies due east at 500 mph while the other plane flies due west at 600 mph. How long will it be before the planes are 3300 miles apart?
4. Two planes leave from the same city at the same time. One plane flies west at 475 mph and the other plane flies east at 550 mph. How long will it be before the planes are 4100 miles apart?
5. Two boaters leave a boat ramp traveling in opposite directions. The first boat travels 15 mph faster than the second one. If after 4 hours, the boats are 220 miles apart, how fast are the boats traveling?
6. Bus A leaves Eastport at 7 a.m. at a speed of 40 km/h and is bound for Westport. Bus B leaves Eastport at 7.45 a.m. and drives along the same freeway at 50 km/h. Bus C leaves Westport at 7.45 a.m. traveling towards Eastport along the same freeway at 60 km/h. Westport and Eastport are 220 km apart. Will bus B or bus C pass bus A first?

Round Up

As long as you learn the *formulas for speed, distance, and time*, then these problems are just like solving the rest of the real-life problems in this Section. As always, first *set up an equation*, then *solve*.

Topic 2.6.1

Section 2.6 Investment Tasks

California Standards:

15.0: Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

What it means for you:

You'll learn what annual interest is, and you'll solve real-life problems involving interest on investment.

Key words:

- annual interest
- rate
- investment

Interest is the money you earn by investing — that's why it's also known as "return on investment."

Money can be invested in lots of ways (for example, savings accounts, stocks, property, etc.), and each can have a different rate of interest.

Annual Interest is the Return After One Year

If your money is invested in a plan that pays interest at a rate of r per year, then the compound interest formula tells you the value of your savings after t years.

$$A = p(1 + r)^t$$

A = amount including interest t = length of investment in years
 p = initial amount invested r = annual interest rate

Putting $t = 1$ shows that after one year, you will have $A = p + pr$.

But p is the original amount you invest (called the principal), and so the amount of interest earned in a single year (I) is equal to pr .

Annual interest: $I = pr$

Example 1

Maria invests \$5000 in a savings account with an annual interest rate of 10%. What is the return on her investment at the end of one year?

Solution

The formula you need is $I = pr$, where $p = \$5000$.

Convert the percent to a decimal: $r = \frac{10}{100} = 0.10$

Substitute in your values for p and r :

$$\begin{aligned} I &= pr = 5000 \times 0.10 \\ &= \mathbf{500} \end{aligned}$$

So Maria makes **\$500** in interest on her investment.

✓ Guided Practice

1. A banker invested \$7000 at an annual interest rate of 8%.
What would be the return on the investment at the end of the year?
2. A banker invested \$5000 at an annual rate of 2.5%.
What would be the return on the investment at the end of one year?
3. A banker invested \$10,000 at an annual rate of 13.25%.
What would be the return on the investment at the end of one year?
4. At the end of one year an investment at 6% earned \$795.00 in interest. What amount was invested?
5. At the end of one year an investment at 3.5% earned \$28.00 in interest. What amount was invested?
6. At the end of one year an investment of \$1250 earned \$31.25 in interest. What was the interest rate?
7. At the end of one year an investment of \$6000 earned \$630 in interest. What was the interest rate?

You Could Invest in Two Schemes at Once

If you invest your money in two different schemes, the **total interest** over one year is the **sum of the interest** earned from each of the different schemes.

$$I = p_1 r_1 + p_2 r_2$$

where p_1 is the amount invested in Scheme 1 at an interest rate of r_1 , and p_2 is the amount invested in Scheme 2 at an interest rate of r_2 .

Example 2

Francis has \$10,000 to invest for one year. He plans to invest \$6000 in stocks, and put the rest in a savings account. If the stocks pay 10% annually and the savings account pays 8%, how much interest will he make over the year?

Solution

$$I = p_1 r_1 + p_2 r_2 \quad \text{where } p_1 = 6000, r_1 = 0.1 (= 10\%) \\ p_2 = 10,000 - 6000 = 4000, r_2 = 0.08 (= 8\%)$$

$$\begin{aligned} I &= p_1 r_1 + p_2 r_2 \\ &= (6000 \times 0.10) + (4000 \times 0.08) \\ &= 600 + 320 \\ &= \mathbf{920} \end{aligned}$$

So Francis will earn **\$920** over the year.

✓ Guided Practice

8. Tyler invested \$12,000 in two different savings accounts for one year. One account had \$8000 and paid a 10% return annually and the other had \$4000 and paid a 12.5% return. How much interest did he earn over the year?
9. Maria invested some money in two different stock accounts for one year. If she invested \$5000 in an account with an annual return of 2.5%, how much would she have had to invest at 3% in order to receive \$260 interest for the year?
10. Yang is investing the same amount of money into two different savings accounts. He invests in an account returning 10% and an account returning 12% for one year. If he earned \$1226.50 in interest, how much did he invest in each account?

Some Problems Require You to Work Backwards

Sometimes you're given information about **total return on investment** — and you have to work backwards to figure out how money was invested in the first place.

Example 3

A banker invested \$12,000 for one year. He invested some of this money in an account with an interest rate of 5%, and the rest of the money in stocks which paid 9% interest annually.

How much money did he invest in each plan if the total return from the investments was \$700?

Solution

Call p_1 the amount invested at 5%, and p_2 the amount invested at 9%.

Let $p_1 = x$, then $p_2 = 12,000 - x$.

Also, $r_1 = 0.05$ (= 5%), and $r_2 = 0.09$ (= 9%).

Substitute these values into the annual interest formula:

$$I = p_1 r_1 + p_2 r_2 = 0.05x + 0.09(12,000 - x) = 700$$

Now you can solve for x :

$$0.05x + 0.09(12,000 - x) = 700$$

$$5x + 9(12,000 - x) = 70,000$$

Get rid of decimal coefficients

$$5x + 108,000 - 9x = 70,000$$

$$5x - 9x = 70,000 - 108,000$$

Group like terms

$$-4x = -38,000$$

Simplify

$$x = 9500$$

So he invested **\$9500 in the account paying 5%**, and $12,000 - 9500 =$ **\$2500 in stocks**.

✓ Guided Practice

11. Latisha invested a total of \$60,000. She invested a part of her money at an annual interest rate of 6% and the rest at 10%. If the total return at the end of the year was \$5200, how much was invested at each rate?
12. Robin invested \$80,000. He invested some of his money at an annual interest rate at 10% and the rest at 12%. If the total interest earned at the end of one year was \$8300, how much was invested at each rate?
13. Leon invests \$15,000 in two different accounts. He invests some at a 3% interest rate and the remaining at 4.5%. If he earns \$621.00 in interest for one year, how much did he invest at each rate?
14. Fedder invested some money in three different savings accounts. If he invested \$7500 in an account with an annual return of 8% and \$4000 in an account earning 8.9%, how much money, to the nearest dollar, did he invest at 8.25% if he earned \$1166.30 total interest in a year?

Sometimes Two Amounts of Interest are Related

In Example 4, instead of relating the **sum** of the individual amounts of interest to a given **total**, you have to relate them to **each other**.

Example 4

A banker had \$80,000 to invest. She put some of the money in a deposit account paying 10% a year, and invested the rest at 8%. The annual return on the money invested at 8% was \$100 more than the return on the 10% investment. How much money did she invest at each rate?

Solution

Call p_1 the amount invested in the deposit account, and p_2 the amount invested at 8%.

Let $p_1 = x$, then $p_2 = 80,000 - x$.

Also $r_1 = 0.1$ (= 10%), and $r_2 = 0.08$ (= 8%).

This time, you know the return from the 8% plan was \$100 more than the return from the 10% plan. Writing this as an equation gives:

$$p_1 r_1 + 100 = p_2 r_2$$

Substitute all the information into this formula, then solve for x :

$$0.1x + 100 = 0.08(80,000 - x)$$

$$10x + 10,000 = 8(80,000 - x)$$

$$10x + 10,000 = 640,000 - 8x$$

$$18x = 630,000$$

$$x = 35,000$$

So she invested **\$35,000 at a rate of 10%**,
and $80,000 - 35,000 =$ **\$45,000 at a rate of 8%**.

Guided Practice

15. Dorothy divided \$14,500 among two accounts paying 11% and 8% interest annually. The interest earned after 1 year in the 8% account was \$455 less than that earned in the 11% account. How much money was invested in each account?
16. Michael invested \$18,000 in two different accounts. The account paying 14% annually earned \$1357.10 more than the one earning 15% interest. How much was invested in each account?
17. Lavasha invested \$16,000 among two different accounts paying 10% and 12% in one year. If she earned twice as much interest in the account paying 12%, how much did she invest in each account?

Independent Practice

1. In one year, an investment at 8% interest earned \$1060. How much money was invested?
2. Louise invested \$25,000 in two different accounts. She invested some money at a 4.5% interest rate and the remaining amount at a 3.25% interest rate. If she earned \$1062.50 in interest in a year, how much did she invest at each interest rate?
3. Mackey invested some money in an account paying 6% interest for one year. She invested \$1000 more than this amount in an account paying 8.5%. How much did she invest in total if the total interest earned in the year was \$737.50?
4. Maxwell invested a total of \$100,000. He invested his money in three different accounts. He invested \$35,000 at 10% annual interest rate, \$42,000 at 8% annual interest rate, and the remainder at 14% interest rate. How much interest did Maxwell receive in one year?
5. Garrett invested \$12,000 at an annual interest rate of 6%. How much money would Garrett have had to invest at 10% so that the combined interest rate for both investments was 7% over the year?
6. Two business partners decided to borrow \$30,000 start-up money for their business. One borrowed a certain amount at a 16.5% interest rate and the other one borrowed the rest of the money at a 12.5% interest rate. How much money did each business partner borrow if the total interest amount at the end of the year was \$4150?

Round Up

Any problems that involve adding together investments with different *interest rates* are actually a type of *mixture problem*. In the next Topic you'll deal with mixture problems not involving money.

Mixture Tasks

California Standards:

15.0: Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

What it means for you:

You'll set up and solve equations involving mixtures.

Key words:

- mixture task
- volume
- mass
- concentration
- ingredients

Check it out:

Concentration means “amount per liter,” or “amount per kilogram,” etc.

The interest problems in Topic 2.6.1 were actually mixture problems because you had to add together returns from investments with different interest rates. In this Topic you'll see some mixture problems that don't involve money.

Some Tasks Use “Amount per Unit Mass/Volume”

Mixtures are made by combining **ingredients**. In math, mixture problems involve using algebra to work out the **precise amounts** of each ingredient. To do this, you have to make use of an equation of the form:

$$\text{concentration} = \frac{\text{amount of substance}}{\text{total volume (or total mass)}}$$

The “amount of substance” could be given to you as a **volume** or as a **mass** — but it doesn't make any difference to the math. Just use whatever units they give you in the question.

Concentration can also be described as a percent:

$$\text{percent of ingredient} = \frac{\text{amount of ingredient}}{\text{total weight or volume of mixture}} \times 100$$

The following example doesn't involve a mixture, but it shows how the above formula can be used.

Example 1

If a 1 kg bag of granola consists of 15% raisins (by mass), how many grams of raisins are there in the bag?

Solution

The formula tells you:

$$\text{Percent of raisins} = \frac{\text{total amount of raisins}}{\text{total amount of granola}}$$

$$0.15 = \frac{\text{total amount of raisins}}{1000}$$

So total mass of raisins = $1000 \times 0.15 = 150$ grams

✓ Guided Practice

1. A 19-ounce can of a disinfectant spray consists of 79% ethanol by weight. How many ounces of pure ethanol does the spray contain?
2. A 2 kg bag of mixed nuts contains 20% walnuts (by mass). How many grams of walnuts are in the bag?
3. What is the percent of juice if a 2 liter bottle of juice drink has 250 ml of juice?
4. A 142 g bottle of baby powder contains 15% zinc oxide. How many grams of zinc oxide are in the powder?
5. What is the concentration of a hydrogen peroxide solution if 500 ml of the solution contains 15 ml of hydrogen peroxide?

You Can Calculate the Concentration as a Percent

This example shows how you can apply the formula to **mixtures** of various concentrations.

Example 2

If you combine 5 liters of a 10% fruit juice drink and 15 liters of a 20% fruit juice drink, what percent of fruit juice do you have?

Solution

Here, you have to use your formula more than once, but the principle is the same.

$$\text{Concentration of fruit juice} = \frac{\text{volume of fruit juice}}{\text{volume of fruit drink}}$$

For the **10% fruit drink**, concentration = $0.1 = \frac{x}{5}$, where x is the volume of fruit juice.

The volume of fruit juice in the 10% solution is $x = 0.1 \times 5 = 0.5$ liters.

And for the **20% fruit drink**, concentration = $0.2 = \frac{y}{15}$, where y is the volume of fruit juice.

The amount of fruit juice in the 20% solution is $y = 0.2 \times 15 = 3$ liters.

Also, the **total volume** of the combined fruit drink is $5 + 15 = 20$ liters.

The **total amount** of fruit juice in the mixture is $x + y = 0.5 + 3 = 3.5$ liters.

Therefore, the concentration of the final mixture is:

$$\text{Concentration} = \frac{\text{volume of fruit juice}}{\text{volume of fruit drink}} = \frac{3.5}{20} = 0.175 = \mathbf{17.5\%}$$

Guided Practice

6. If you combine 10 liters of juice cocktail made with 100% fruit juice with 5 liters of another juice cocktail containing 10% fruit juice, what percent of the 15 liters of juice cocktail will be fruit juice?
7. There is 4% hydrogen peroxide in a 475 ml hydrogen peroxide solution. The 4% solution is mixed with 375 ml of a hydrogen peroxide solution containing 15% hydrogen peroxide. What percent of hydrogen peroxide is in the combined solution?
8. Tina mixes 4.75 mg of a lotion containing 1% vitamin E with 2 mg of a lotion containing 7% vitamin E. What is the percentage of vitamin E in the combined lotion?

Independent Practice

1. A 12 fl. oz. bottle of lotion contains 2.5% lavender. How many fluid ounces of lavender are contained in the bottle of lotion?
2. A 2.5 fl. oz. bottle of nose spray contains 0.65% sodium chloride. What volume of sodium chloride does the spray contain?
3. A 5-gallon car radiator should contain a mixture of 40% antifreeze and 60% water. What volumes of water and antifreeze should the radiator contain?
4. A 1.75 ml bottle of medicine contains 80 mg of an active ingredient per 0.8 ml of medicine. How many milligrams of active ingredient does the bottle contain?
5. A 473 ml bottle of rubbing alcohol contains 331.1 ml of isopropyl alcohol. What percentage of the bottle is isopropyl alcohol?
6. Two liters of 100% pure pineapple juice is mixed with 2 liters of soda water and 1 liter of orange juice to make a party punch. What percentage of the party punch is pineapple juice?
7. A 2 kg bag of walnuts is mixed with 3 kg of pecans, 4 kg of hazelnuts, and 1 kg of brazil nuts to make a 10 kg bag of mixed nuts. What percentage of the 10 kg bag of nuts is pecans and hazelnuts?
8. 473 ml of 70% rubbing alcohol is combined with 473 ml of 90% rubbing alcohol and 473 ml of water. What is the percentage of rubbing alcohol in the mixture?
9. Two liters of juice cocktail are combined with 6 liters of a different juice cocktail to make 8 liters of juice cocktail containing 50% juice. If the 6 liters of juice cocktail contains $33\frac{1}{3}\%$ fruit juice, what percentage of fruit juice is the 2 liters of juice cocktail?

Round Up

*It's a good idea to **memorize** the two formulas at the start of this Topic — knowing them will make it a lot easier to deal with any **mixture tasks** involving **liquids** or **ingredients**.*

More Mixture Tasks

California Standards:

15.0: Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

What it means for you:

You'll set up and solve equations involving mixtures.

Key words:

- mixture task
- volume
- mass
- concentration
- ingredients

Remember that a mixture is made up of only the original ingredients.

Nothing can mysteriously appear in the final mixture that was not part of the original ingredients, and nothing can disappear either.

The Total Amount of Each Ingredient Doesn't Change

$$\begin{array}{l} \text{total amount of each} \\ \text{substance in the ingredients} \end{array} = \begin{array}{l} \text{amount of that substance} \\ \text{in the final mixture} \end{array}$$

So if you know the amount of a substance in **each of the ingredients** you are mixing, you can always calculate the amount of that substance in the **final mixture**.

Similarly, if you know the amount of a substance in one ingredient and in the final mixture, you can find out how much was in the other ingredient(s).

Example 1

There are 1000 gallons of water in a wading pool. The water is 5% chlorine. What quantity of a 65% chlorine solution would need to be added to bring the pool's chlorine concentration up to 15%?

Solution

Ingredient 1: You know the volume and concentration of the water in the wading pool.

Volume = 1000 gallons. Concentration of chlorine = 0.05 (= 5%).

So the amount of chlorine in the wading pool to begin with is given by:

$$\begin{aligned} \text{Original amount of chlorine} &= \text{volume} \times \text{concentration} \\ &= 1000 \times 0.05 = \mathbf{50} \end{aligned}$$

Ingredient 2: You also know that the concentration of the chlorine solution to be added is 0.65 (or 65%). So if you call the volume of the added chlorine solution x , then:

The amount of chlorine added is $0.65x$ (= volume \times concentration)

Mixture: This means the final volume of water in the wading pool is $1000 + x$, and so the final amount of chlorine in the pool is $0.15(1000 + x)$.

Therefore $50 + 0.65x = 0.15(1000 + x)$.

Now solve for x : $50 + 0.65x = 150 + 0.15x$

$$0.5x = 100$$

$$x = 200$$

Therefore **200 gallons of the 65% solution** are required.

✓ Guided Practice

1. A pest control company has 200 gallons of 60% pure insecticide. To create a 70% pure insecticide solution, the company must mix the 60% pure insecticide solution with a 90% pure insecticide solution. How many gallons of the 90% pure insecticide solution should the company mix with the existing solution?
2. A chemist needs to dilute a 60% citric acid solution to a 20% citric acid solution. She needs 30 liters of the 20% solution. How many liters of the 60% solution and water should be used? (Hint: water has 0% acid.)
3. An alloy containing 40% gold is mixed with an 80% gold alloy to get 1000 kilograms of an alloy that is 50% gold. How many kilograms of each alloy are used?

There's a Formula for the Amount of Each Substance

Multiplying the **volume** by the **concentration** gives you the amount of substance in each solution. And since the total amount of substance is the same before and after, you can use this handy formula:

$$c_1v_1 + c_2v_2 = cv$$

where c_1 and v_1 are the concentration and volume of the first ingredient,
 c_2 and v_2 are the concentration and volume of the second ingredient,
and c and v are the concentration and volume of the mixture of the two.

The next example is quite similar to Example 1, but it uses the formula shown above.

Example 2

A pharmacist has 500 cm³ of a 30% acid solution. He replaced x cm³ of the 30% solution with a 70% acid solution to get 500 cm³ of a new 40% acid solution. What volume of the 30% acid solution did he replace with the 70% acid solution?

Solution

Assume that the pharmacist poured away x cm³ of the 30% solution and replaced it with 70% solution.

$$\text{Then } 0.3(500 - x) + 0.7x = 0.4(500)$$

$$150 + 0.4x = 200$$

$$x = 125$$

So the pharmacist replaced **125 cm³ of the 30% acid solution**.

✓ Guided Practice

4. A doctor prescribes 20 grams of a 62% solution of a generic medicine. The pharmacist has bottles of 50% and 70% solutions in stock. How many grams of each solution can the pharmacist use to fill the prescription for the patient?
5. Fifty pounds of special nuts costing \$4.50 per pound were mixed with 120 pounds of generic nuts that cost \$2 per pound. What is the value of each pound of the nut mixture?
6. Cherry juice that costs \$5.50 per liter is to be mixed with 50 liters of orange juice that costs \$2.50 per liter. How much cherry juice should be used if the value of the mixture is to be \$3.50 per liter?
7. Sixty liters of a syrup that costs \$8.50 per liter were mixed with honey that costs \$4.75 per liter. How many liters of honey were used if the value of the mixture is \$6 per liter?

Always Relate the “Before” and “After” Totals

Example 3

Marie ordered 40 pounds of walnuts at \$1.50 per pound. She mixed this with hazelnuts costing \$1.00 per pound, and sold the mixed nuts at \$1.25 per pound.

How many pounds of hazelnuts did she use, if she broke even?

Solution

Here, your “amount per unit weight” formula is:

$$\text{Price per pound } (p) = \frac{\text{total value } (v)}{\text{weight } (w)}$$

First write down what you know about the original ingredients and the final mixture (calling the quantity you need to find x , for example).

Then relate a “before” total to the “after” total — so the total value of the ingredients is the same as the total value of the mixture — $v_1 + v_2 = v$.

\$1.50-per-pound walnuts: weight = **40** lb. Total value = $1.50 \times 40 = \mathbf{60}$.

\$1.00-per-pound hazelnuts: weight = x lb. Total value = $1.00 \times x = \mathbf{x}$.

Mixed (\$1.25) nuts: weight = **(40 + x)** lb. Total value = **1.25(40 + x)**.

To break even, the value of the mixed nuts must equal the sum of the values of the original nuts. That is, $x + 60 = 1.25(x + 40)$.

$$\begin{aligned}\text{Solve this to find } x: \quad & x + 60 = 1.25(x + 40) \\ & x + 60 = 1.25x + 50 \\ & -0.25x = -10 \\ & \mathbf{x = 40}\end{aligned}$$

Therefore she used **40 pounds of hazelnuts**.

Check it out:

Notice that $p_1w_1 + p_2w_2 = pw$. Compare this to the formula on the previous page — it’s essentially the same.

✓ Guided Practice

8. Mrs. Roberts owns a pet store and wishes to mix 4 pounds of cat food worth \$2.00 per pound with another cat food costing \$3.00 per pound. How much of the \$3.00 per pound cat food should be used if the mixture is to have cost \$2.75 per pound?
9. A local garden center makes a medium grade plant seed by mixing a low grade plant seed bought for \$2.50 a pound with 14 pounds of superior quality seed bought for \$5.00 a pound. How much of the low grade plant seed needs to be mixed with the superior quality seed if it is to be worth \$3.50 per pound?
10. A tea blend is made by using 2 kg of \$2.00 per kg tea leaves and another tea leaf costing \$4.00 per kilogram. How many kilograms of the \$4.00 tea is needed to make a tea blend worth \$3.75 per kilogram?
11. A landscaper wants to make a blend of grass seed using 300 pounds of \$0.40 per pound grass seed and another seed costing \$0.75 per pound. How much of the \$0.75 seed does the landscaper need to make a \$0.60 per pound blend?

A Different Context — but the Same Math

Example 4

A retailer mixed two fruit drinks. He mixed 30 liters of a fruit drink that cost him \$1.90 per liter with an unknown amount of a fruit drink that cost him \$2.50 per liter. If the mixture's ingredients cost \$2.40 per liter, what volume of the \$2.50-per-liter drink did he use?

Solution

This time, use: price per liter (p) = $\frac{\text{total cost } (c)}{\text{volume } (v)}$

\$1.90-per-liter drink: volume = **30** liters. Total cost = $1.90 \times 30 = 57$.

\$2.50-per-liter drink: volume = x liters. Total cost = **$2.50x$** .

Mixed (\$2.40) drink: volume = **$(30 + x)$** liters. Total cost = **$2.40(30 + x)$** .

But the cost of the mixture is the sum of the ingredients' costs, so $57 + 2.50x = 2.40(30 + x)$.

Solving this for x gives:

$$57 + 2.5x = 72 + 2.4x$$

$$0.1x = 15$$

$$x = 150$$

So he used **150 liters of the \$2.50-per-liter fruit drink**.

✓ Guided Practice

12. A pet store owner mixed two different bird seeds. She mixed 20 pounds of a seed that costs \$2.25 per pound with 50 pounds of another seed. If the mixture of seeds cost \$2.75 per pound, how much per pound did the 50 pounds of seed cost?
13. Apples cost \$0.80 per pound and grapes cost \$1.10 per pound. Michael wishes to make a fruit tray using only apples and grapes that costs \$1.00 per pound. If Michael has 8 lbs of apples, how many pounds of grapes are needed?
14. A roast coffee blend costing \$6.60 per pound is made by mixing a bean that costs \$2.00 per pound with another one that costs \$7.00 per pound. If 5 pounds of the \$2.00 bean are used, how many pounds of the \$6.60 beans would be produced?

✓ Independent Practice

1. Dave and his parents went to the movies. Adult tickets cost \$7.00 and children's tickets cost \$3.00. For the screening of one movie, 500 tickets were sold for a total of \$2000. Find the number of each kind of ticket that was sold for the movie.
2. A 70% salt solution was diluted with purified water to produce about 50 liters of 55% salt solution. Approximately how much purified water was used?
3. A chemist wants to dilute a 60% boric acid solution to a 15% solution. He needs 30 liters of the 15% solution. How many liters of the 60% solution and water must the chemist use?
4. A student wants to dilute 35 liters of a 35% salt solution to a 16% solution. How many liters of distilled water does the student need to add to the 35% salt solution to obtain the 16% salt solution?
5. A tea blend is made by mixing 2 kg of a \$2.00 per kg tea with 5 kg of another tea. If the total cost of the ingredients in the blend is \$24.00, what is the price per kg of the 5 kg of tea?
6. How many liters of a 100% alcohol solution must be mixed with 20 liters of a 50% solution to get a 70% solution?
7. A chemist has 8 liters of a 30% solution of a compound. How much of a 100% solution of the compound must be added to get a 50% solution?
8. Milk that contains 2% fat is mixed with milk containing 0.5% fat. How much 0.5% fat milk is needed to be added to 10 gallons of 2% fat milk to obtain a mixture of milk containing 1% fat?

Round Up

Over the last couple of Topics you've seen lots of examples of *mixture tasks*. The most difficult part of a mixture task is *setting up the equation* — once you've done that it's all a lot easier.

Topic 2.7.1

Section 2.7 Work Rate

California Standards:

15.0: Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

What it means for you:

You'll solve combined work rate problems by calculating each work rate in turn.

Key words:

- work rate
- combined work rate

Check it out:

Problems like this usually assume that both people are working all the time (rather than one of them stopping early).

Work problems are similar to the mixture problems you saw in Section 2.6. Again, the only new thing is that there are a few formulas that you need to know in order to set up the equations.

Work-Related Tasks are About Speeds of Working

This area of math is concerned with calculating how long certain jobs will take if the people doing the job are working at different rates.

Example 1

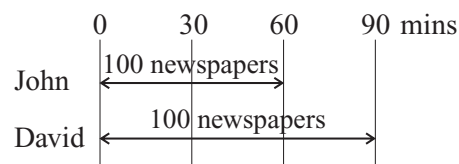
John takes 1 hour to deliver 100 newspapers, and David takes 90 minutes to deliver 100 newspapers. How long would it take them to deliver 100 newspapers between them?

Assume that they work independently, that they both start at the same time, and that they are both working the whole time.

Solution

It's tempting just to work out how long it takes John and David to deliver half the papers each. But that does not take into account the fact that they are working as a team — and since **John works faster** than David, he will **deliver more papers**. Instead, you have to figure out **how quickly** they work **as a team** — **not just as two individuals**.

The problem is that you don't know how many papers each of them delivers — you only have the information given in the question, which is shown in this diagram:



You need to work out their **rate** of delivering the papers.

Based on how long it takes each person to deliver 100 papers, you can calculate **how many** papers are delivered **every minute** in **total**.

If John can deliver 100 newspapers in 1 hour (60 minutes),

he can deliver $100 \div 60 = \frac{5}{3}$ newspapers in 1 minute.

If David can deliver 100 newspapers in 90 minutes,

he can deliver $100 \div 90 = \frac{10}{9}$ newspapers in 1 minute.

So in total $\frac{5}{3} + \frac{10}{9} = \frac{15}{9} + \frac{10}{9} = \frac{25}{9}$ newspapers are delivered each minute.

This means that 100 papers will take $100 \div \frac{25}{9} = \mathbf{36 \text{ minutes to deliver}}$.

✓ Guided Practice

1. Akemi can weed the garden in 4 hours. Keira can weed the garden in 12 hours. How long would it take the two of them to weed the garden together?
2. Martha can clean a statue in 15 hours. Chogan can clean the same statue in 9 hours. If they clean the statue together, how long would it take them to finish?
3. A carpenter can build a cabinet in 10 hours. Her assistant can build the same cabinet in 15 hours. How long would it take them to build the cabinet together, assuming they can work independently?

Work Rate = Work Done ÷ Time

Work rate is the amount of work carried out per unit time.

The **work completed** can be given as a **fraction**. For example, if only half the task is completed, write $\frac{1}{2}$. If the whole task is completed, write 1.

$$\text{Work rate} = \frac{\text{work completed}}{\text{time}}$$

When you are **solving a problem** like this, you need to start by **identifying** the “**work completed**” and the **time** that it took. The “work completed” part in the previous example was quite straightforward. The example below is not so simple — make sure you understand each step.

Example 2

An inlet pump can fill a water tank in 10 hours. However, an outlet pump can empty the tank in 12 hours. An engineer turns on the inlet pump but forgets to switch off the outlet pump. With both pumps running, how long does it take to fill the tank?

Solution

$$\text{Inlet pump's work rate} = \frac{\text{work completed}}{\text{time}} = \frac{1}{10}$$

$$\text{Outlet pump's work rate} = \frac{\text{work completed}}{\text{time}} = \frac{1}{12}$$

The two pumps are working in **opposite directions**.

The **combined work rate** is the difference between the two rates.

$$\text{So the combined rate} = \frac{1}{10} - \frac{1}{12} = \frac{6 \cdot 1}{60} - \frac{5 \cdot 1}{60} = \frac{6-5}{60} = \frac{1}{60}$$

So after 60 hours, the net amount of water that has gone in is 1 tank's worth — which means the tank will be full. So the answer is **60 hours**.

Check it out:

The inlet pump's “work completed” is 1 tank of water **in**, and the outlet pump's “work completed” is 1 tank of water **out**.

Check it out:

The fractions have been subtracted using the least common multiple (LCM) of 10 and 12, which is 60. See Topic 2.3.1 for more on LCMs.

Guided Practice

4. Megan, Margarita, and James work in a fast-food restaurant after school. It takes Megan 6 hours, Margarita 3 hours, and James 4 hours to clean the utensils individually. How long would it take the three of them to clean the utensils if they worked together?
5. A bathtub can be filled from a faucet in 10 minutes. However, a pump can empty the bathtub in 15 minutes. If the faucet and the pump are on at the same time, how long will it take to fill the bathtub?
6. Isabel can fence the family three-acre lot in 8 days. If it would take José 6 days and Marvin 12 days to fence the same lot, how long would it take the three people to fence the lot together?

Independent Practice

1. Bill can paint a house in 5 days and Samantha can paint the same house in 7 days. How long will it take them, working together, to paint the house?
2. There are two drains in a tub filled with water. One drain would empty the tub in 3 hours, if opened. The other drain would empty the tub in 4 hours. If both drains are opened at the same time, how long will it take to empty the tub?
3. A tub has 2 drains. One drain can empty the full tub in 20 minutes. The other drain can empty half the tub in 30 minutes. How long will it take to empty the full tub if both drains are opened together?
4. Jose can paint a house in 4 days. Leroy can paint the same house in 5 days. How long will it take Jose and Leroy, working together, to paint the house?
5. Sam, Doris, and Alice are weeding a yard. Working by himself, Sam could weed the yard in $1\frac{1}{2}$ hours. Doris could do the same yard by herself in 1 hour and Alice could do it in only 45 minutes. How long will it take the three of them working together to weed the yard?
6. A barrel has two filling pipes and one draining faucet. One pipe could fill the barrel in 2 hours and the other could fill it in $2\frac{1}{2}$ hours. The faucet could empty the barrel in 4 hours. How long will it take to fill the barrel if both pipes are filling and the draining faucet is opened?

Round Up

*The very first thing you need to do with a **work rate problem** is look through the word problem to identify the **work completed** and the **time taken**.*

Combined Work Rate

California Standards:

15.0: Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

What it means for you:

You'll solve combined work rate problems by first calculating the combined work completed.

Key words:

- work rate
- combined work rate

In Topic 2.7.1 you calculated each work rate in turn, then added or subtracted them.

This Topic contains a method for *combining* each piece of information directly into *one equation* — which can make the whole calculation a lot more straightforward.

Combined Work Rates Include All the Information

Another way to approach work rate problems is to put all the information from the question directly into this equation, which can then be solved:

$$\text{Combined work rate} = \frac{\text{combined work completed}}{\text{total time}}$$

Here's the same problem that you saw in Example 2 in Lesson 2.7.1 — but now using the new method.

Example 1

An inlet pump can fill a water tank in 10 hours. However, an outlet pump can empty the tank in 12 hours. An engineer turns on the inlet pump but forgets to switch off the outlet pump. With both pumps running, how long does it take to fill the tank?

Solution

$$\text{Combined work rate} = \frac{\text{total water in the tank}}{\text{time taken to fill the tank}}$$

Let x = number of hours to fill the tank, and then substitute everything you know into the formula:

$$\frac{1}{10} - \frac{1}{12} = \frac{1}{x} \quad \begin{array}{l} \leftarrow \text{1 tank's worth of water} \\ \leftarrow \text{total time} \end{array} \quad \begin{array}{l} \text{(left-hand side of equation} \\ \text{= combined rate)} \end{array}$$

Then rearrange to solve:

$$\begin{aligned} 60x \left(\frac{1}{10} - \frac{1}{12} \right) &= \frac{60x}{x} \\ \frac{60x}{10} - \frac{60x}{12} &= \frac{60x}{x} \\ 6x - 5x &= 60 \\ \mathbf{x} &= \mathbf{60} \end{aligned}$$

So it takes **60 hours** to fill the tank.

Check it out:

The fractions have been multiplied by the LCM of 10, 12, and x , to get rid of the fractional coefficients.

Example 2

Jesse can paint a wall in 6 hours. Melinda can paint the same wall in 4 hours. How long would it take the two of them to paint the wall together if they worked independently and started at the same time?

Solution

Jesse can complete the **whole** task in 6 hours, so her work rate = $\frac{1}{6}$.

Melinda can complete the **whole** task in 4 hours, so her work rate = $\frac{1}{4}$.

You need to find out how many hours they would take to do the work **together**. They are working on the same wall, so **add** the contribution from each person:

$$\text{Combined work rate} = \frac{1}{6} + \frac{1}{4}$$

Use x for the hours it takes in total, and write out the equation for combined work rate.

$$\frac{1}{6} + \frac{1}{4} = \frac{1}{x}$$

$$12x \left(\frac{1}{6} + \frac{1}{4} \right) = \frac{12x}{x}$$

Multiply by $12x$, the LCM of 6, 4, and x

$$2x + 3x = 12$$

Then solve for x

$$5x = 12$$

$$x = 2\frac{2}{5}$$

Therefore painting the wall would take Jesse and Melinda

$2\frac{2}{5}$ hours = 2 hours and 24 minutes.

Example 3

Liza can dig a garden in 7 hours alone. If Marisa helps her, they finish all the digging in just 3 hours, working independently. How long would it take Marisa to dig the garden alone?

Solution

Liza can complete the **whole** task in 7 hours, so her work rate = $\frac{1}{7}$.

Together they can complete the **whole** task in 3 hours, so their

combined work rate = $\frac{1}{3}$.

Example 3 continued

You need to find out how fast Marisa could do the work **on her own**.

Use x for the hours it takes Marisa, so her work rate = $\frac{1}{x}$.

This time you know the combined work rate, but not Marisa's work rate.

The formula is still: combined work rate = $\frac{\text{combined work completed}}{\text{total time}}$.

$$\frac{1}{7} + \frac{1}{x} = \frac{1}{3}$$

Write out the equation

$$21x\left(\frac{1}{7} + \frac{1}{x}\right) = \frac{21x}{3}$$

Multiply by $21x$, the LCM of 7, x , and 3

$$3x + 21 = 7x$$

Then solve for x

$$4x = 21$$

$$x = 5\frac{1}{4}$$

So Marisa can dig the garden in $5\frac{1}{4}$ hours = **5 hours and 15 minutes**.

Guided Practice

1. A pump can fill a fuel tank in 30 minutes. A second pump can fill the same tank in 60 minutes. How long would it take to fill the fuel tank if both pumps were filling the tank together?
2. Machine A can pack 50 crates of canned dog food in 15 minutes. When Machine A and Machine B are working at the same time, they can pack 50 crates of canned dog food in 9 minutes. How long would it take Machine B to pack 50 crates of canned dog food by itself?
3. Three student-service workers are cataloging books in a school library. Joaquin can catalog the books in 8 hours, Caroline can do the same job in 4 hours, and Joshua can do it in 6 hours. If the three students work together, how long will it take them to finish cataloging the books?
4. A central heater can warm a house in 24 minutes. When the central heater and a floor heater are used together they can warm the house in 16 minutes. How long would the floor heater take to warm the house alone?
5. When Dionne's cell phone has been fully charged, it can be used for 5 hours before its battery runs out. When its battery runs out, the phone can be fully charged in 3 hours if it is not in use. Dionne's phone's battery has run out. If she sets it to recharge, and uses the phone constantly while it is recharging, how long will it take to fully charge her phone?

Independent Practice

1. A window cleaner can clean the windows of a house in 5 minutes and his trainee can clean the same windows in 20 minutes. How long would it take the window cleaner and his trainee to clean the windows together?
2. Susan and John are hired to stuff envelopes with parent notices about an upcoming school event. If the task would take Susan 120 minutes and John 90 minutes individually, how long would it take the two of them to stuff the envelopes together?
3. Lorraine can tile a room in 20 hours. Juan can tile the same room in 30 hours, and Oliver can tile it in 40 hours. How long would it take them to tile the room if they worked together?
4. A faucet can fill a barrel in 2 hours. However, there is a hole in the bottom of the barrel that can empty it in 6 hours. Without knowing of the hole, Leo tries to fill the barrel. How long will it take?
5. Tyrone, Jerry, and Dorothy are painting a fence surrounding a field. Tyrone could paint the fence in 4 days by himself. Jerry could paint the same fence, but it would take him 6 days if he did it by himself. Dorothy could paint the fence in $3\frac{1}{2}$ days. How long should it take the three of them, working together, to paint the fence?
6. Michael can decorate the cafeteria for a dance in 4 hours. Emily can decorate the cafeteria in 3 hours. Eylora can decorate the cafeteria in $1\frac{1}{2}$ hours. If the three of them have 1 hour, working together, to decorate for the party, will it be done in time?
7. When turned on, a faucet can fill a tub in 3 hours. The tub has 2 drains — the first can empty the tub in 8 hours and the second can empty the tub in 7 hours. If the faucet is turned on while both drains are open, how long will it take to fill the tub?
8. Manuel and Anita have extensive gardens. Manuel can mow all the lawns with his lawn mower in 5 hours. If Anita helps him with her lawn mower, they can mow all the lawns in 2 hours. How long would it take Anita to mow all the lawns alone with her lawn mower?
9. Albert and Po work in a factory that makes leotards. Albert cuts out the material, and Po stitches the side seams. Albert can cut the parts for 1 box of leotards in 9 hours, and Po can stitch the side seams for 1 box of leotards in 27 hours (including any breaks). If Albert and Po start work at the same time, how long will it be before there is 1 full box of leotards cut and waiting for their side seams to be stitched?

Round Up

*These work rate problems often include a lot of information, and it's easy to get the values mixed up. It's a good idea to **check** that your solutions make sense in the **original word problem**.*

Topic 2.8.1

Section 2.8

Absolute Value Equations

California Standards:

3.0: Students solve equations and inequalities involving absolute values.

What it means for you:

You'll solve equations involving absolute values.

Key words:

- absolute value

Check it out:

In other words, an absolute value is never negative.

You already saw *absolute values* on the number line in Topic 1.2.3.

In this Topic you'll see that absolute values can turn up in equations too — and you'll learn how to get rid of them to *solve* the equations.

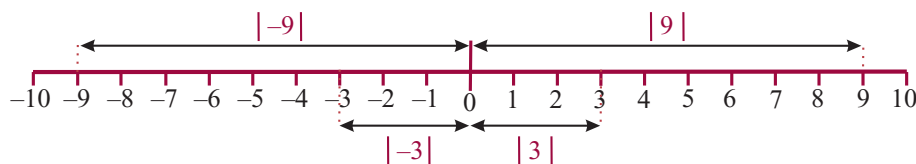
The Absolute Value is Always Positive

The absolute value of x is defined as:

x if x is positive or zero, and $-x$ if x is negative.

The absolute value of any real number, x , is written as $|x|$.

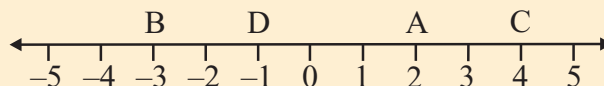
Another way to think about it is that the absolute value is the distance of a number from zero on the number line. For example, the distance between 0 and 9 is the **same** as the distance between 0 and -9 . This can be written as $|9| = |-9| = 9$.



The distance between 0 and 3 is the same as the distance between 0 and -3 . This can be written as $|3| = |-3| = 3$.

✓ Guided Practice

Find the distance that each letter is from zero:



1. A
2. B
3. C
4. D

Simplify:

5. $|-9|$
6. $|-20|$
7. $-|-2|$
8. $-|-7|$

Absolute Value Equations Have Two Solutions

Watch out when you're solving an equation such as $|2x| = 10$.

There are actually **two** numbers whose distance from zero (their absolute value) is 10: **-10 and 10**.

So, the equation $|2x| = 10$ can be rewritten as **two** separate equations:
 $2x = 10$ or $2x = -10$

Here's the same information about absolute values written in math-speak:

Let $c \geq 0$. If $|x| = c$, then $x = c$ or $x = -c$.

Example 1

Solve $|3x| = 12$.

Solution

There are two separate equations to solve:

$$3x = 12 \quad \text{or} \quad 3x = -12$$

Solve each equation for x .

$$\frac{3x}{3} = \frac{12}{3}$$

$$x = 4$$

$$\frac{3x}{3} = -\frac{12}{3}$$

$$x = -4$$

Check your answers by substituting back into the original equation.

$$|3(4)| = 12$$

$$|12| = 12$$

$$12 = 12 \quad \checkmark$$

$$|3(-4)| = 12$$

$$|-12| = 12$$

$$12 = 12 \quad \checkmark$$

So $x = 4$ and $x = -4$ are the solutions of the equation.

Independent Practice

Solve:

1. $|4x| = 16$

2. $|-2x| = 8$

3. $|-8x| = 24$

4. $|3x| = 9$

5. $|4.8x| = 144$

6. $|0.02x| = 9$

7. $|-1.04x| = 0.2392$

8. $|2x + 1x| = 171$

9. $|10x - 5x| = 100 + 5$

Write these as absolute value equations and find the solutions:

10. The product of four and a number is a distance of 20 from 0.

11. A number has a distance of 8 from 0.

12. Twice a number has a distance of 0.6 from 0.

13. The product of 1.6 and a number has a distance of 0.304 from 0.

Round Up

The main thing to remember with absolute values in an equation is that they result in **two possible solutions**. In the next Topic you'll work through more complicated equations involving absolute values.

More on Absolute Value

California Standards:

3.0: Students solve equations and inequalities involving absolute values.

What it means for you:

You'll solve equations involving absolute values.

Key words:

- absolute value

This Topic is quite similar to the last one — but this time the absolute values have *more than one term*. That means there's a little more solving to do once you've *removed* the absolute value signs.

Absolute Values May Include More than One Term

If there's more than just a single term in the absolute value signs, you need to keep those terms together until the absolute value signs have been removed.

Example 1

Solve $|2x - 5| = 7$.

Solution

Rewrite this as two separate equations:

$$2x - 5 = 7 \quad \text{or} \quad 2x - 5 = -7$$

Solve both equations for x :

$$\begin{aligned} 2x &= 12 \\ x &= 6 \end{aligned}$$

$$\begin{aligned} 2x &= -2 \\ x &= -1 \end{aligned}$$

Check your answers by substituting back into the original equation.

$$\begin{aligned} |2(6) - 5| &= 7 \\ |12 - 5| &= 7 \\ |7| &= 7 \\ 7 &= 7 \quad \checkmark \end{aligned}$$

$$\begin{aligned} |2(-1) - 5| &= 7 \\ |-2 - 5| &= 7 \\ |-7| &= 7 \\ 7 &= 7 \quad \checkmark \end{aligned}$$

So $x = 6$ and $x = -1$ are the correct solutions of the equation.

Guided Practice

Find all possible solutions to these absolute value equations:

1. $|12 - 4x| = 18$

2. $|2x - 8| = 4$

3. $|5x - 3x - 1| = 10$

4. $|3j + 1| = 10$

5. $\left| -\frac{3}{g} \right| = 15$

6. $\left| -\frac{8}{x} \right| = 1.6$

Find all possible solutions to these equations when $a = -4$:

7. $|2x + a| = 10$

8. $|ax + 8| = 144$

9. $|3x + 8| = a$

10. $|ax - a| = -a$

You Might Have to Rearrange the Equation First

If the absolute value is not alone on one side of the **equals sign**, the equation must be **rearranged** to get the **absolute value on its own** before the equation can be split into two parts.

To solve an equation of the form $d|ax + b| + c = k$, rewrite the equation in the form $|ax + b| = \frac{k - c}{d}$, then solve for x .

Example 2

Solve $2|3x - 1| + 4 = 12$.

Solution

Rearrange the equation to get the absolute value on its own:

$$2|3x - 1| + 4 = 12$$

$$2|3x - 1| = 8$$

$$\frac{2|3x - 1|}{2} = \frac{8}{2}$$

$$|3x - 1| = 4$$

Now split this into the two parts:

$$3x - 1 = 4$$

$$3x = 5$$

$$x = \frac{5}{3}$$

$$3x - 1 = -4$$

$$3x = -3$$

$$x = -1$$

Check your answers:

$$2\left|3\left(\frac{5}{3}\right) - 1\right| + 4 = 12$$

$$2|5 - 1| + 4 = 12$$

$$2|4| + 4 = 12$$

$$8 + 4 = 12 \quad \checkmark$$

$$2|3(-1) - 1| + 4 = 12$$

$$2|-4| + 4 = 12$$

$$8 + 4 = 12 \quad \checkmark$$

So $x = \frac{5}{3}$ and $x = -1$ are the correct solutions of the equation.

✓ Guided Practice

Find all possible solutions to these equations involving absolute values:

11. $|2y - 7| + 5 = 20$

12. $11 + |-7 - 2k| = 21$

13. $|x - 9| - 5 = 1$

14. $-11 + |5 - 4y| = 2$

15. $4|3y - 4| = 8$

16. $-2|6 - 5y| = -8$

17. $-3|12 - 7x| = -6$

18. $7 - 2|8 - 4x| = -9$

19. $\left|\frac{2y-3}{3}\right| = 8$

20. $\left|\frac{3x-4}{5}\right| = 7$

21. $\left|\frac{5y-1}{6}\right| - 3 = 5$

22. $10 + \left|\frac{2x-5}{4}\right| = 11$

You Might Have Absolute Values on Both Sides

When it comes to solving equations with absolute values on **both sides**, you end up with **four** equations rather than two:

Example 3

Solve $|3x - 2| = |4 - x|$.

Solution

$$\begin{aligned} |3x - 2| &= |4 - x| \\ \pm(3x - 2) &= \pm(4 - x) \end{aligned}$$

There are four possible solutions:

- (1) $3x - 2 = 4 - x$
- (2) $3x - 2 = -(4 - x)$
- (3) $-(3x - 2) = 4 - x$
- (4) $-(3x - 2) = -(4 - x)$

But — this set of four equations only contains **two different equations**:

- if you take equation (4) and divide both sides by -1 , you get equation (1).
- if you take equation (3) and divide both sides by -1 , you get equation (2).

So if there are absolute values on both sides of an equation, then you can treat one of them as though it is not an absolute value — so writing $|3x - 2| = |4 - x|$ is equivalent to writing $|3x - 2| = 4 - x$ or $3x - 2 = |4 - x|$.

More generally:

$$\begin{aligned} |ax + b| = |cx + d| &\text{ is equivalent to} \\ |ax + b| = cx + d &\text{ or } ax + b = |cx + d|. \end{aligned}$$

Example 4Solve $|5x - 2| = |10 - x|$.**Solution**Rewrite the equation with only one absolute value: $|5x - 2| = 10 - x$

Write out two separate equations to solve.

$$5x - 2 = +(10 - x)$$

$$5x - 2 = -(10 - x)$$

Solve the equations for x .

$$5x - 2 = 10 - x$$

$$5x - x - 2 = -10 + x - x$$

$$5x + x - 2 = 10 - x + x$$

$$4x - 2 = -10$$

$$6x - 2 = 10$$

$$4x - 2 + 2 = -10 + 2$$

$$6x - 2 + 2 = 10 + 2$$

$$4x = -8$$

$$6x = 12$$

$$x = -2$$

$$x = 2$$

Check your answers by substituting them into the original equation.

$$|5x - 2| = |10 - x|$$

$$|5x - 2| = |10 - x|$$

$$|5(2) - 2| = |10 - (2)|$$

$$|5(-2) - 2| = |10 - (-2)|$$

$$|10 - 2| = |8|$$

$$|-10 - 2| = |10 + 2|$$

$$|8| = |8|$$

$$|-12| = |12|$$

$$8 = 8 \quad \checkmark$$

$$12 = 12 \quad \checkmark$$

So $x = 2$ and $x = -2$ are the correct solutions of the equation. **Guided Practice**

Find all possible solutions to these absolute value equations:

23. $|2x - 8| = |3x - 12|$

24. $|5x - 7| = |3x + 15|$

25. $|9x + 18| = |4x - 2|$

26. $|6x - 13| = |15 - 8x|$

27. $|4x - 36| = |2x - 4|$

28. $|8x + 4| = |12x - 2|$

29. The distance of $(-2x - 8)$ from 0 is 3. What are the possible values of x ?**30.** If $(2x + 4)$ and $(3x + 8)$ are the same distance from 0, what are the possible values of x ?**31.** If $(x + 8)$ is the same distance from 0 as $(4x - 8)$, what are the possible values of x ?If $c = 10$, find all possible solutions to these equations:

32. $|2x - c| = |c - x|$

33. $|cx - 5| = |x - c|$

34. $|4x - c| = |2x + c|$

35. $|2cx + 4| = |x - 3c|$

✓ Independent Practice

Solve:

1. $|2x| = 84$

2. $|3z| = -9$

3. $|x + 8| = 24$

4. $|-x + 4| = 3$

5. $|2x + 8| = |4 - 3x|$

6. $|7x - 4| = |3x + 9|$

7. $|0.1x - 0.3| = |0.3x + 4.1|$

8. $-2|x - 20| = -8$

9. $\left|\frac{1}{3}x + \frac{1}{8}\right| = \left|\frac{1}{12}x + \frac{1}{4}\right|$

10. $\left|-\frac{x}{4}\right| = 12$

11. $\left|\frac{1}{2}x\right| + 1 = 5$

12. $\frac{4}{3}|2x + 1| = 8x$

13. $\left|\frac{x - 8}{0.1}\right| = \left|\frac{10(x - 4)}{3}\right|$

14. $\left|\frac{x - 4}{7}\right| = \left|\frac{2x + 8}{14}\right|$

In Exercises 15–18 you will need to form an absolute value equation and solve it to find the unknown.

15. If $(x + 4)$ is $3x$ from 0, what are the possible values of x ?

16. If $(4x - 5)$ is $(2x + 1)$ from 0, what are the possible values of x ?

17. If $(3w + 2)$ and $(w - 4)$ are the same distance away from 0, what are the possible values of w ?

18. If $(4x - 5 + x)$ and $(7 + 5x + 2)$ are the same distance from 0, what are the possible values of x ?

19. Given that $|3x - 5| = |2x + 6|$, find the two possible values of $b^2 - 2bx + x^2$ if $b = -3$.

If $a = 2$, $b = 4$, and $c = 6$ then solve each absolute value equation for x :

20. $|ax - b| = |x + c|$

21. $|ax + b| + c = a - bx$

22. $\frac{1}{a}|cx + ab| = |ax - c|$

Round Up

Equations with *absolute values on both sides* look really difficult at first. Make sure you understand how Example 3 shows that you still only get *two distinct equations* when there are *two absolute values*.

Chapter 2 Investigation

Wildlife Park Trains

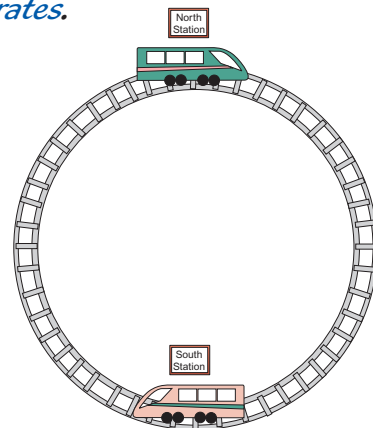
This Investigation is all about writing and solving equations about *rates*.

A large wildlife park is circular in shape and has a diameter of 7 miles. Around the outskirts of the park is a circular train track.

Two automatic trains travel around the park, both in a **clockwise** direction. The trains are designed to both average 20 mph so that they never meet. However, one train has developed a fault and now travels at 18 mph.

The trains set off from stations on opposite sides of the park at **9 a.m.** At what time will the faster train **catch up** with the slower train?

(Assume that the trains instantly reach their average speeds and that you can ignore the lengths of the trains.)



Things to think about:

- What is the **distance** around the track?
- When the trains meet, the faster train will have traveled **further** than the slower train. How much further? If the slower train has traveled **x miles**, how far has the faster train traveled?
- When they get to the meeting point, they will have both been traveling for the same amount of time. **How long** will each train have been traveling for in terms of x ?

$$\text{Circumference} = \pi \times \text{diameter. Use } \pi = \frac{22}{7}.$$

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

Extension

- 1) **Where** will the trains be when one catches up with the other?
How many times will each train pass its starting point before they meet?
- 2) When the faster train has caught up with the slower train, it **changes direction**. Both trains are now traveling in opposite directions. After **how many minutes** will the trains meet again?

Open-ended extension

The park manager wants the trains to run each day while the park is open **without** one train catching up with the other. Unfortunately, the speeds of the trains **cannot** be changed for technical reasons. The opening times of the park are shown on the right.

Write a **report** to the manager recommending how he can achieve this. You may wish to include diagrams of the track in your report.



Round Up

Parts of this Investigation are tough because the trains are traveling in a circle — but the main thing with any real-life problem is to **write down all the math** carefully before you start solving.

Chapter 3

Single Variable Linear Inequalities

Section 3.1	Inequalities	137
Section 3.2	Applications of Inequalities	147
Section 3.3	Compound Inequalities	155
Section 3.4	Absolute Value Inequalities	158
Investigation	Mailing Packages	162

Topic 3.1.1

Section 3.1 Inequalities

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.

What it means for you:

You'll go over inequality notation from grade 7 and you'll show inequalities on the number line.

Key words:

- inequality
- endpoint
- interval

Inequalities work like equations, but they tell you whether one expression is bigger or smaller than the expression on the other side.

In grade 7 you solved linear inequalities — so the stuff in the next couple of Topics should be fairly familiar.

Inequality Symbols

Expressions such as $3x > 8$, $x < -5$, $x \geq 10$, and $x \leq 10$ are inequalities.

An inequality is a mathematical sentence that states that two expressions are not equal. You read the inequality symbols like this:

$<$ “is less than”
 $>$ “is greater than”
 \leq “is less than or equal to”
 \geq “is greater than or equal to”

For example, you read $m < c$ as “ m is less than c ” and you read $k \geq 5$ as “ k is greater than or equal to 5.”

You Can Show Inequalities on the Number Line

The inequality $x > 4$ represents the **interval** (part of the number line) where the numbers are **greater than 4**.

Similarly $k \geq 5$ represents all real numbers **greater than or equal to 5** on the number line.

The numbers 4 and 5 in these examples represent the **endpoints of the intervals** of the number line under consideration. However, $k \geq 5$ **includes the endpoint 5**, while $x > 4$ **excludes the endpoint 4**.

Example 1

Show the inequality $x > 4$ on the number line.

Solution

You can show $x > 4$ on the number line like this:



Note that the **circle at the endpoint is open** — this shows that the **endpoint 4 is not part of the interval** of the number line defined by the inequality $x > 4$.

This is called an **open interval**, denoted as $(4, \infty)$.

Check it out:

The symbol ∞ and the closing parenthesis indicate that the list of numbers continues indefinitely (which is also shown by the arrow on the graph).

Check it out:

The “[” bracket indicates that the endpoint is part of the interval, which is also shown by the closed circle in the graph.

Check it out:

Note also that the interval is closed at one end, but open at the other — which is why it's called a half-open interval.

Example 2

Show the inequality $k \geq 5$ on the number line.

Solution



Note that the **closed circle** at 5 indicates that the **endpoint 5** is included in the interval.

This is called a **half-open interval**, denoted as $[5, \infty)$.

Guided Practice

Write each of the following inequalities in interval notation, and show each graphically.

1. $k > 6$

2. $m < -2$

3. $x \geq 1$

4. $j > -5$

5. $n \leq 3.5$

6. $l \leq 3$

7. $3 \geq y$

8. $j \geq 10$

9. $10 \leq x$

10. State the inequality represented on the number line opposite.



11. State the inequality represented on the number line opposite using interval notation.



Independent Practice

In Exercises 1–3 write each inequality in interval notation.

1. $r > 8$

2. $t \leq -9$

3. $3 \leq x$

In Exercises 4–6 write each interval as an inequality in x .

4. $(-\infty, 2)$

5. $(-6, \infty)$

6. $(-\infty, 0]$

In Exercises 7–12 show each inequality or interval graphically.

7. $k > -7$

8. $k \leq 2$

9. $1.5 \leq y$

10. $(-11, \infty)$

11. $[0, \infty)$

12. $(-\infty, 5]$

13. Anthony is shopping for a birthday gift for his cousin Robert. He has \$25 in his wallet. Write an inequality that shows how many dollars he can spend on the gift.

14. Teresa is only allowed to swim outside if the temperature outside is at least 85 °F. Write an inequality that shows the temperatures in degrees Fahrenheit at which Teresa is allowed to swim.

15. In order to achieve an ‘A’ in math, Ivy needs to score more than 95% on her next test. Write an inequality that shows the test score Ivy needs to achieve in order to earn her ‘A’ in math.

Round Up

The most difficult thing is remembering that a “[” bracket shows the endpoint is included in the interval, and a “(” parenthesis means the endpoint isn't included.

Addition and Subtraction Properties of Inequalities

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.

5.0: Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

What it means for you:

You'll solve inequalities that contain + and - signs.

Key words:

- inequality
- endpoint
- interval

Check it out:

The circle is open at the endpoint 7, which shows that 7 isn't part of the solution interval.

To solve inequalities such as $x + 1 > 4$, $3x + 2 > 7$, or $x - 7 \leq 12$, you need to apply *properties of inequalities* — they're just like the properties of equality you applied to equations.

Addition Property of Inequalities

Addition Property of Inequalities

Given real numbers a , b , and c , if $a > b$, then $a + c > b + c$.

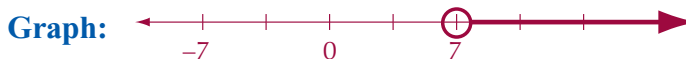
In other words, adding the same number to both sides of an inequality gives an equivalent inequality.

Example 1

Solve and graph the solution of $x - 2 > 5$ on a number line. Write the solution in interval notation.

Solution

$$\begin{aligned} \text{Solve: } \quad x - 2 &> 5 \\ x - 2 + 2 &> 5 + 2 && \text{Using addition property of inequalities} \\ x &> 7 \end{aligned}$$



Solution in interval notation: $(7, \infty)$

Guided Practice

Solve and graph each inequality. Write each solution set in interval notation.

1. $l - 1 \geq 3$

2. $r - 3 < -5$

3. $x - 4 \leq -1$

4. $m - 7 > -10$

5. $-2m < 2 - 3m$

6. $-k \leq -2k + 1$

7. $-2j + 3 \geq -j$

8. $-7(j + 1) < -6j$

Subtraction Property of Inequalities

Subtraction Property of Inequalities

Given real numbers a , b , and c , if $a > b$, then $a - c > b - c$.

That is, subtracting the same number from both sides of an inequality gives an equivalent inequality.

Example 2

Solve and then graph the solution of $3x \leq 6 + 2x$ on a number line. Write the solution in interval notation and then state the maximum integer value of x that satisfies the inequality.

Solution

$$\begin{aligned}\text{Solve: } \quad 3x &\leq 6 + 2x \\ 3x - 2x &\leq 6 + 2x - 2x && \text{Subtraction property of inequalities} \\ x &\leq 6\end{aligned}$$



Solution in interval notation: $(-\infty, 6]$
So the **maximum integer value of x is 6**.

Check it out:

In this example, the half-open interval closes with a bracket to show that the endpoint 6 is part of the solution interval.

Some Problems Need You to Use Both Properties

Example 3

Solve and graph the solution set of $5x - 2 \leq 4x - 3$ on a number line. State the maximum integer value of x that satisfies the inequality, and write the solution set in interval notation.

Solution

$$\begin{aligned}\text{Solve: } \quad 5x - 2 &\leq 4x - 3 \\ 5x - 2 + 2 &\leq 4x - 3 + 2 && \text{Addition property of inequalities} \\ 5x &\leq 4x - 1 \\ 5x - 4x &\leq 4x - 4x - 1 && \text{Subtraction property of inequalities} \\ &&& \text{and commutative property} \\ x &\leq -1\end{aligned}$$



Solution in interval notation: $(-\infty, -1]$
So the **maximum integer value of x is -1**.

Check it out:

The closed circle of the graph shows that the endpoint is included in the solution interval.

✓ Guided Practice

Solve and graph each inequality, and write each solution set in interval notation. Find also the maximum or minimum integer value that satisfies each inequality.

9. $x + 2 < 8$

11. $j + 2 \geq -3$

13. $3k + 2 \geq 2k$

15. $2t + 1 \leq t - 10$

10. $t + 5 \leq 6$

12. $y + 3 > -9$

14. $4x - 3 < 3x$

16. $6(x - 1) \geq 5(x + 2)$

✓ Independent Practice

Solve the following inequalities.

Justify each step and write each solution set in interval notation.

1. $j - 3 > -1.5$

3. $x + 3 < -11$

5. $5k - 1.25 < 4k - 4$

2. $k - 2 \geq -5$

4. $2v - 2.5 > v + 1$

6. $-7j - 5 < -6j$

Solve and graph each inequality in Exercises 7–18.

Write each solution set in interval notation.

7. $6x - 1 \geq 5x + 3$

9. $4(x - 3) - 3x < -11$

11. $7t + 6(1 - t) \leq -4$

13. $2(t + 5) \leq 3t - 11$

15. $5(-2 + 3x) \geq -2(1 - 7x)$

17. $13 + 3(x - 5) \leq 2(-3 + x)$

8. $3(k - 2) - 2k \geq 9$

10. $4n - 3(n + 1) \geq 1$

12. $5t - 4(t - 1) > -3$

14. $-3x + 2 + 7x < -3 + 3x - 4$

16. $-4(1 - 4x) \geq -7.5(-1 - 2x)$

18. $5x - (3x - 4) > -(2 - x)$

19. Find the maximum integer value of x if $6x + 4 \leq 5x - 8$.

20. Find the maximum integer value of x if $-3(3 - 2x) < 5(x - 5)$.

21. Find the least integer value of x if $-2(1 - x) > x - 1$.

22. Find the least integer value of x if $4(x - 3) \leq 5x + 2$

23. Stephen needs to buy a new uniform for soccer. He already has \$25, but the uniform costs \$55. He participates in car washes to help pay for the uniform. Write an inequality to represent the amount of money, x , that Stephen needs to earn from the car washes in order to be able to afford the new uniform. Use this inequality to find the minimum amount of money he needs to earn.

24. An art gallery sells Peter's paintings for \$ x , and keeps \$100 commission. This means Peter is paid $\$(x - 100)$ for each painting. If Peter wants to make at least \$750 for a particular painting, write an inequality to represent the amount, x , that the gallery needs to sell that painting for. Use this inequality to find the minimum price of the painting.

Round Up

*Adding and subtracting with inequalities is a lot like dealing with normal equations — so there's nothing in this Topic that should cause you too much trouble. Next up is **multiplying** and **dividing**.*

Multiplication and Division Properties of Inequalities

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.

5.0: Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

What it means for you:

You'll solve inequalities that contain \times and \div signs.

Key words:

- inequality
- reverse

After a Topic on addition and subtraction with inequalities, you know what to expect: *multiplication* and *division* of inequalities.

Multiplication Property of Inequalities

Multiplication Property of Inequalities

Given real numbers a , b , and c , if $a > b$ and $c > 0$ then $ac > bc$.

That is, **multiplying both sides** of an inequality by a **positive number** gives an **equivalent inequality**. For example:

Start with an inequality: $4 < 10$...is true.

Multiplying by 2 gives: $2 \times 4 < 2 \times 10$

$8 < 20$...which is also true.

Example 1

Solve $\frac{1}{5}x > 2$.

Solution

$$\frac{1}{5}x > 2$$

$$5 \times \frac{1}{5}x > 5 \times 2$$

Multiplication property of inequalities

$$x > 10$$

Guided Practice

Solve each inequality in Exercises 1–12.

1. $\frac{1}{2}x \leq 3$

2. $\frac{1}{4}p > 12$

3. $\frac{1}{8}a \geq 4$

4. $6 > \frac{1}{3}x$

5. $\frac{x}{4} > -5$

6. $\frac{1}{5}k > \frac{1}{15}$

7. $\frac{j}{9} \leq 12$

8. $\frac{1}{3} < \frac{1}{9}a$

9. $\frac{k}{12} \geq \frac{1}{4}$

10. $\frac{y}{20} \leq \frac{-1}{5}$

11. $\frac{2}{5}p < 4$

12. $-6 < \frac{3}{4}a$

Division Property of Inequalities

Division Property of Inequalities

Given real numbers a , b , and c , if $a > b$ and $c > 0$ then $\frac{a}{c} > \frac{b}{c}$.

In other words, **dividing both sides** of an inequality by a **positive number** gives an **equivalent inequality**. For example:

Start with an inequality: $4 < 10$...is true.
Dividing by 2 gives: $4 \div 2 < 10 \div 2$
 $2 < 5$...which is also true.

Example 2

Solve $3x < 11$.

Solution

$$3x < 11$$

$$\frac{3x}{3} < \frac{11}{3}$$

Division property of inequalities

$$x < \frac{11}{3}$$

✓ Guided Practice

Solve each inequality in Exercises 13–26.

13. $10x < 5$

14. $4y \geq 20$

15. $32 \geq 8n$

16. $7z \leq 28$

17. $40 < 8y$

18. $56 > 7j$

19. $72 \geq 9j$

20. $6k \leq 48$

21. $20k < 5$

22. $7 < 77x$

23. $4 > 28y$

24. $40a < 5$

25. $6(6a) \geq 4$

26. $9 \leq 3b(20 + 7)$

Multiplication Property with Negative Numbers

This is really important — so make sure you read this carefully.

You've just seen that multiplying both sides of an inequality like $4 > 3$ by, say, 3, gives a true new inequality, $12 > 9$.

However, if the original inequality is multiplied by a **negative number** like -3 , the resulting inequality is $-12 > -9$, which is **false**. To make the resulting inequality true, you have to **reverse the inequality sign**. That gives $-12 < -9$, which is true.

Check it out:

Watch out — the difference between this rule and the normal multiplication property is the " $c < 0$ then $ac < bc$ " part.

Multiplication Property of Inequalities — Negative Numbers

Given real numbers a , b , and c , if $a > b$ and $c < 0$ then $ac < bc$.

In other words, if you **multiply both sides** of an inequality by a **negative number**, you have to **reverse the inequality symbol** — otherwise the statement will be **false**. For example:

Start with an inequality: $-3 < 8$...is true.
Multiplying by -2 and reversing the inequality sign gives:
 $-2 \times -3 > -2 \times 8$
 $6 > -16$...which is also true.

Example 3

Solve $-\frac{1}{2}x > 2$.

Solution

$$\begin{aligned} -\frac{1}{2}x &> 2 \\ -2 \times -\frac{1}{2}x &< -2 \times 2 && \text{Multiplication property of inequalities} \\ x &< -4 \end{aligned}$$

✓ Guided Practice

Solve each inequality in Exercises 27–37.

27. $\frac{-1}{10}x \leq 7$

28. $\frac{-1}{9}c < 5$

29. $11 < \frac{-1}{2}x$

30. $\frac{-d}{3} > -4$

31. $\frac{-1}{4}a < -12$

32. $-4 \leq \frac{-1}{9}c$

33. $\frac{-j}{7} \geq 8$

34. $\frac{-k}{11} < -1$

35. $\frac{1}{8} \leq \frac{-1}{8}y$

36. $\frac{-1}{9} \cdot \frac{1}{4} > x(4 - \frac{13}{3})$

37. $\frac{1}{4+5 \cdot 10} \geq -\frac{1}{6}g$

Division Property with Negative Numbers

Division Property of Inequalities — Negative Numbers

Given real numbers a , b , and c , if $a > b$ and $c < 0$ then $\frac{a}{c} < \frac{b}{c}$.

In other words, if you **divide both sides** of an inequality by a **negative number**, you have to **reverse the inequality symbol** — otherwise the statement will be **false**.

Start with an inequality: $-3 < 9$...is true.

Dividing by -3 and reversing the inequality sign gives:

$$\frac{-3}{-3} > \frac{9}{-3}$$

$1 > -3$...which is also true.

Example 4

Solve $-13y < 39$.

Solution

$$-13y < 39$$

$$\frac{-13}{-13}y > \frac{39}{-13}$$

$$y > -3$$

Division property of inequalities

Guided Practice

Solve each inequality in Exercises 38–47.

38. $-5x \leq 5$

39. $36 < -9x$

40. $-6j > 48$

41. $-x > -1$

42. $45 \leq -9x$

43. $-77c < -11$

44. $-72y \geq -8$

45. $49 > -7a$

46. $-4y \geq \frac{1}{4}$

47. $\frac{-1}{10} \geq -6c$

✓ Independent Practice

Solve each inequality in Exercises 1–6.

1. $32 < 4h$

2. $9a \leq -45$

3. $\frac{-1}{7}g > 7$

4. $-8c \leq 48$

5. $2 \leq \frac{6h}{5}$

6. $\frac{-5w}{7} < \frac{8}{5}$

7. Solve $\frac{3}{2}y - \frac{2}{3}(1 - 2y) < 5$, and state the largest possible integer value of y .

Solve each inequality in Exercises 8–21.

8. $2(x - 3) - 3(2 - x) > 8$

9. $-4(3 - 2x) > 5x + 9$

10. $3(x - 1) < 7 - 2x$

11. $5(y + 3) - 7y \leq 3(2y + 3) - 5y$

12. $3y - (5y + 4) > 7y + 2(y - 5)$

13. $5(2x - 3) - 3(x - 7) \geq 4(3x + 2) + 2x - 9$

14. $5 - 4(x + 2) \leq 7 + 5(2x - 1)$

15. $7 - 2(m - 4) \leq 2m + 11$

16. $\frac{2x - 5}{3} - \frac{3x + 1}{4} \geq \frac{11}{12}$

17. $\frac{5}{9}(k - 3) \leq \frac{4}{3}(k + 7) - \frac{1}{9}$

18. $0.5(x - 1) - 0.75(1 - x) < 0.65(2x - 1)$

19. $7 - 3(x - 7) \leq 4(x + 5) + 1$

20. $0.35(x - 2) - 0.45(x + 1) \geq 8 + 0.15(x - 10)$

21. $3(2x + 6) - 5(x + 8) \leq 2x - 22$

22. Laura has \$5.30 to spend on her lunch. She wants to buy a chicken salad costing \$4.20 and decides to spend the rest on fruit. Each piece of fruit costs 45¢.

Write an inequality to represent this situation, and then solve it to find how many pieces of fruit Laura can buy.

23. Audrey is selling magazine subscriptions to raise money for the school library. The library will get \$2.50 for every magazine subscription she sells. Audrey wants to raise at least \$250 for the library.

Write and solve an inequality to represent the number of magazine subscriptions, x , Audrey needs to sell to reach her goal.

24. The total cost of food and supplies for a cat is x dollars per year, and medical expenses can be 1.5 times the cost of food and supplies per year. If Maddie can spend no more than \$500 a year on the cat, what is the most that she can spend on food and supplies?

Write and solve an inequality to represent the situation.

Round Up

Multiplication and division with negative numbers can sometimes be difficult. If you multiply by a negative number and forget to reverse the direction of the inequality sign, then your solution will be wrong — so watch out.

Topic 3.2.1

Section 3.2 Multistep Inequalities

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x-5) + 4(x-2) = 12$.
5.0: Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

What it means for you:

You'll solve inequalities involving several steps.

Key words:

- inequality
- isolate

Check it out:

This example uses the addition, subtraction and division properties of inequalities, as well as the associative property.

Inequality problems often involve using more than one of the properties of inequalities that you saw in Topics 3.1.2 and 3.1.3.

The multistep inequalities in this Topic are a little harder than the ones you saw in Section 3.1 — but you still solve them using the same methods.

Multistep Inequalities Combine Lots of Techniques

To simplify and therefore **solve an inequality** in one variable such as x , you need to **isolate the terms in x** on one side and **isolate the numbers** on the other.

It's often easiest to keep the x -terms on the side of the inequality sign where they have a **positive value**.

Example 1

Solve $4x - 7 > 2x$.

Solution

$$4x - 7 > 2x$$

$$4x - 7 + 7 > 2x + 7$$

Eliminate the -7 from the left first

$$4x > 2x + 7$$

$$4x - 2x > 2x + 7 - 2x$$

Then get rid of the $2x$ on the right

$$2x > 7$$

$$\frac{2x}{2} > \frac{7}{2}$$

Divide both sides by 2 to get x on its own

$$x > \frac{7}{2}$$

Guided Practice

1. $5x - 2 \leq 3$

2. $4x - 1 > 2$

3. $-3a - 3 < -9$

4. $5x - 7 \leq 8$

5. $\frac{k}{4} + 6 < 7$

6. $\frac{x}{2} + 9 \leq -2$

7. $\frac{x}{5} - 3 \geq 4$

8. $\frac{j}{3} - 8 \geq 7$

9. $8g - 10 \leq 9g + 4$

10. $4a + 5 \leq 6a + 9$

11. $4c - 9 \geq 5c + 16$

12. $k - 4 \leq 2k + 20$

Break Complicated Inequalities into Smaller Steps

Here's a useful **checklist** for tackling more complicated inequality questions by breaking them down into easier steps:

Check it out:

Instead of "x," the variable might be "y," "z," or any other letter.

Solving Inequalities

1. Multiply out any **parentheses**.
2. **Simplify** each side of the inequality.
3. Remove **number terms** from one side.
4. Remove **x-terms** from the other side.
5. Multiply or divide to get an **x-coefficient of 1**.

Example 2

Solve the inequality: $7(x - 2) - 3(x - 4) > 2(x - 5)$

Solution

$$7x - 14 - 3x + 12 > 2x - 10$$

$$4x - 2 > 2x - 10$$

$$4x - 2 + 2 > 2x - 10 + 2$$

$$4x > 2x - 8$$

$$4x - 2x > 2x - 8 - 2x$$

$$2x > -8$$

$$\frac{2x}{2} > \frac{-8}{2}$$

$$x > -4$$

Multiply out the parentheses

Simplify

Eliminate the -2 from the left

Get rid of the 2x on the right

Divide to get x on its own

✓ Guided Practice

13. $4x - 2 \leq 3(x + 5)$

15. $4x + 1 > 2(x + 2)$

17. $8(x + 3) \leq 7(x + 3)$

19. $\frac{6(x + 3)}{5} > 3x$

21. $\frac{4(x + 3)}{3} \leq 3(x - 2)$

23. $2(x - 1) \geq 4(x - 2) - 8$

14. $6x - 8 > 5(x + 2)$

16. $5x - 4 < 3(x + 6)$

18. $2(x + 1) \geq 4(x - 2)$

20. $\frac{3(x - 2)}{4} < 3x$

22. $\frac{5(x - 2)}{2} \geq 10(x + 1)$

24. $12(b + 1) - 10b > 7(b + 3) + 6$

Independent Practice

In Exercises 1–7, solve each inequality.

1. $9x - 7 \leq 11$

2. $8c - 10 \geq 7c + 6$

3. $6x - 12 > 5x + 8$

4. $11(x + 8) \geq 12x - 12$

5. $9(p + 5) > 10p - 5$

6. $\frac{a}{2} < a + 4$

7. $\frac{a}{6} - 3 \geq 5$

In Exercises 8–28, solve each inequality and write the solution set in interval notation.

8. $6x - 2 \leq 4(x + 5)$

9. $3(x + 1) < 5x + 5$

10. $5x + 1 > 3(x + 3)$

11. $8(x - 1) \geq 4x - 4$

12. $6(j - 2) > 7(j + 4)$

13. $3(x + 2) \geq 5(x - 2)$

14. $\frac{7(a - 4)}{2} \leq 4a$

15. $\frac{4(x + 1)}{6} > 2x$

16. $\frac{7(x - 7)}{3} < 2(x + 4)$

17. $\frac{8(x + 3)}{6} \leq 3(x - 2)$

18. $3(k - 1) + 2(k + 1) < 4$

19. $7(d + 3) + 2(d - 4) > -5$

20. $6(x - 4) - 5(x + 1) \leq 9$

21. $4(t + 0.25) - 8(t - 7) \geq -3$

22. $4(x - 1) \geq 8(x - 2) - 6x$

23. $-5(t - 2) + 4 > -(t + 2)$

24. $4(t + 0.5) \leq 0.5(4t - 12) + 6$

25. $5(a - 1) - 2a < 4(a + 4)$

26. $4(a + 3) + 4 \geq -(a - 1)$

27. $6(t + 1) - 10 > 7(t + 3) + 4t$

28. $5(a - 5) - 3 > 4(a + 8) + 7a$

Round Up

If you ever get stuck when you're solving inequalities with *more than one step*, refer to the checklist on the previous page. Just take it *one step at a time*, as if you were dealing with an equation.

Applications of Inequalities

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.

5.0: Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

What it means for you:

You'll solve real-life problems involving inequalities.

Key words:

- inequality
- isolate

Check it out:

These rules are pretty much the same as when you're dealing with real-life equation problems.

Some real-life problems include phrases like “at least” or “at most,” or deal with *maximums* or *minimums*. If you come across these phrases, chances are you'll need to model the situation as an *inequality*.

Applications of Inequalities are “Real-Life” Problems

In the same way that applications of equations are real-life problems, applications of inequalities are **real-life** inequalities problems.

Real-life problems involving inequalities could be about pretty much anything — from finding the area of a field to figuring out how many CDs you can buy with a certain amount of money.

What they all have in common is that they'll all be **word problems** — and you'll always have to **set up and solve an inequality**.

Solving Real-Life Inequality Problems

1. First decide how you will label the variables...
2. ...then write the task out as an inequality...
3. ...make sure you include all the information given...
4. ...then solve the inequality.

Example 1

Find the three smallest consecutive even integers whose sum is more than 60.

Solution

First you need to **label** the variables:

Let x = first (smallest) even integer

$x + 2$ = next (second) even integer

$x + 4$ = next (third) even integer

Check it out:

In this example, you know you need a “>” and not a “≥” because the problem says “more than,” not “more than or equal to.”

Example 1 continued

Then you need to write it out as an inequality:

In English: The sum of three consecutive even integers is more than 60.

In math: $x + (x + 2) + (x + 4) > 60$

Then simplify:

$$x + x + 2 + x + 4 > 60$$

$$3x + 6 > 60$$

$$3x + 6 - 6 > 60 - 6$$

$$3x > 54$$

$$\frac{3x}{3} > \frac{54}{3}$$

$$x > 18$$

Answer in math: $x > 18$

Answer in English: The smallest even integer is more than 18.

So the three smallest consecutive even integers whose sum is greater than 60 are **20**, **22**, and **24**.

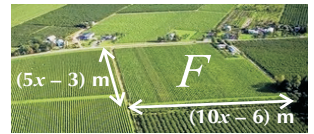
 **Guided Practice**

1. Find the largest three consecutive odd integers whose sum is at most 147.
2. Find the three smallest consecutive odd integers whose sum is more than 45.
3. Find the smallest three consecutive odd integers such that the sum of the first two integers is greater than the sum of the third integer and 11.
4. Find the smallest three consecutive even integers whose sum is greater than 198.
5. The difference between a number and twice that number is at least 7. Find the smallest possible integer that satisfies this criterion.
6. Three times a number is added to 11, and the result is less than 7 plus twice the number. Find the highest possible integer the number could be.
7. José scored 75 and 85 on his first and second algebra quizzes. If he wants an average of at least 83 after his third quiz, what is the least score that José must get on the third quiz?
8. Lorraine’s test scores for the semester so far are 60%, 70%, 75%, 80%, and 85%. If the cutoff score for a letter grade of B is 78%, what is the least score Lorraine must get on the final test to earn a B?

Example 2

The perimeter of a rectangular field, F , is given by the formula $p = 2l + 2w$.

As shown in the diagram, the length $l = (10x - 6)$ m and the width $w = (5x - 3)$ m.



Find the possible values of x for which the given rectangular field would have a perimeter of at least 1182 meters.

Solution

The variables are **already labeled**, so you just need to write it out as an **inequality**:

Substituting the variables into the expression for the perimeter gives:

$$p = 2(10x - 6) + 2(5x - 3)$$

You're also told that the perimeter is at least 1182 meters — in math, you write that as $p \geq 1182$.

So the inequality is:

$$\begin{aligned} 2(10x - 6) + 2(5x - 3) &\geq 1182 \\ 20x - 12 + 10x - 6 &\geq 1182 \\ 30x - 18 &\geq 1182 \\ 30x - 18 + 18 &\geq 1182 + 18 \\ 30x &\geq 1200 \\ \frac{30x}{30} &\geq \frac{1200}{30} \\ \mathbf{x} &\geq \mathbf{40} \end{aligned}$$

Answer in math: $x \geq 40$

Answer in English: x would have to be greater than or equal to 40.

Guided Practice

9. Lily wants to build a fence along the perimeter of her rectangular garden. She cannot afford to buy more than 14 meters of fencing. The length of the garden, l , is $(2x - 3)$ m and the width, w , is $(3x - 10)$ m. Write an inequality to represent the situation and solve it. What would the dimensions of the garden fence have to be to keep the fencing no longer than 14 meters.

10. A car travels $(17x - 5)$ miles in $(x - 7)$ hours. The car travels at a constant speed not exceeding the speed limit of 55 miles per hour. If $\text{speed} = \text{distance} \div \text{time}$, write an inequality to represent this situation and find the minimum possible number of miles traveled.

11. The formula for calculating the speed of an accelerating car is $v = u + at$, where v is the final speed, u is the original speed, a is the acceleration, and t is the time taken. Car A starts at 5 m/s and accelerates at 4 m/s². At the same time, car B starts at 10 m/s and accelerates at 2 m/s². Write and solve an inequality to find out how long it will be before Car A is traveling faster than Car B.

Example 3

A long-distance telephone call from Los Angeles, California, to Harare, Zimbabwe, costs \$9.50 for the first three minutes, plus \$0.80 for each additional minute (or fraction of a minute). Colleen has \$18.30 to spend on a call. What is the maximum number of additional minutes she can spend on the phone?

Solution

First you need to **label** the variables:

Let x = the additional number of minutes after the first 3 minutes.

Then you need to write it out as an inequality:

In English: The total cost is \$9.50 plus \$0.80 per additional minute.
The total cost must be less than or equal to \$18.30.

In math: $9.50 + 0.80x = \text{Total cost for the call}$
 $\text{Total cost for the call} \leq 18.30$
 $\therefore 9.50 + 0.80x \leq 18.30$

Then simplify:

$$\begin{aligned} 950 + 80x &\leq 1830 \\ 950 - 950 + 80x &\leq 1830 - 950 \\ 80x &\leq 880 \\ \frac{80x}{80} &\leq \frac{880}{80} \\ x &\leq 11 \end{aligned}$$

Answer in math: $x \leq 11$

Answer in English: The number of additional minutes must be no more than 11.

So Colleen can spend up to **11 additional minutes** on the phone.

Guided Practice

12. Marisa is buying a new car. Car A costs \$20,000, and has an average annual fuel cost of \$1000. Car B costs \$22,500, and has an average annual fuel cost of \$500. After how many years will Car A have cost more than Car B? Assume that all other maintenance costs are equal for both cars.

13. A cell phone company offers its customers either Plan A or Plan B. Plan A costs \$90 per month with unlimited air time. Plan B costs \$60 per month, plus 50¢ for each minute of cell phone time.

How many minutes can a customer who chooses Plan B use the cell phone before the cost of the calls exceeds the amount it would have cost under Plan A?

Independent Practice

1. Bank T's checking account has monthly charges of an \$8 service fee plus 6¢ per check written. Bank S's checking account has monthly charges of a \$10 service fee plus 4¢ per check written. A company has 150 employees, and pays them monthly by check. The company's financial adviser suggests that Bank S would be cheaper to use. Set up and solve an inequality that supports this recommendation.
2. Jack is doing a sponsored swim to raise money for charity. His mom sponsors him \$10, plus \$1 for every length of the pool he completes. His uncle sponsors him just \$1.50 for every length he completes. How many lengths will Jack have to complete for his uncle to pay more than his mom?
3. On average Sendi uses 350 minutes of air time per month. Company M offers a cell phone plan of \$70 per month plus 85¢ for each minute of air time. Company V offers a cell phone plan of \$130 per month plus 65¢ for each minute of air time. Sendi chooses Company V. Use an inequality to show that this plan is cheaper for her.
4. A group of friends want to drive to a beach resort and spend 5 days there. A car rental firm offers them two rental plans; \$15 a day plus 30¢ per mile traveled, or \$20 a day plus 10¢ per mile. Which rental plan would be better if the beach resort is 150 miles from home, and why?
5. A bank charges a \$10 monthly service fee plus 5¢ handling fee per check processed through its Gold checking account. The bank also offers a Platinum checking account and charges a \$15 monthly service fee plus 3¢ handling fee per check drawn from this account. What is the highest number of checks per month for which the Gold account is cheaper than the Platinum account?
6. A group of executives is traveling to a meeting, so they decide to hire a car and travel together. The car rental agency rents luxury cars at \$65 per day plus 65¢ per mile traveled, or \$55 per day plus 85¢ per mile traveled. What is the maximum number of miles that they can drive before the \$55 per day plan becomes more expensive than the \$65 per day plan?

Round Up

*This Topic is very similar to **real-life applications of equations**, which is covered in Sections 2.4–2.7. Always remember to give your solution as a **sentence** that answers the original problem.*

Topic 3.3.1

Section 3.3

Compound Inequalities

California Standards:

4.0: Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.

5.0: Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

What it means for you:

You'll learn about problems involving two inequalities, and how to solve them.

Key words:

- conjunction
- disjunction
- inequality

Check it out:

The solution set of the conjunction is the numbers for which **both** inequalities are true.

Sometimes a math problem gives you *two different restrictions* on a solution, using *inequality signs*.

A *compound inequality* is two inequalities together — for example, $2x + 1 < 5$ and $2x + 1 > -1$.

Conjunction Problems Include the Word “And”

The word “**and**” means the compound inequality below is a “**conjunction**.” You can rewrite a conjunction as a single mathematical statement, usually involving two inequality signs, like this:

$$2x + 1 < 5 \text{ and } 2x + 1 > -1 \text{ can be rewritten as } -1 < 2x + 1 < 5.$$

✓ Guided Practice

In Exercises 1–9, express each conjunction as a single mathematical statement.

1. $3y - 1 > 5$ and $3y - 1 < 11$
2. $4a + 7 > -10$ and $4a + 7 < -2$
3. $-8c + 2 \leq 16$ and $-8c + 2 \geq -3$
4. $7x - 2 < 14$ and $7x - 2 > 4$
5. $10a - 7 < 2$ and $10a - 7 > -5$
6. $9t + 4 \leq 4$ and $9t + 4 \geq 3$
7. $-4g - 5 < -5$ and $-4g - 5 > -10$
8. $7c - 9 < 7$ and $7c - 9 > -4$
9. $8y + 9 \leq 2$ and $8y + 9 > -6$

Solving Conjunctions

The solution to a conjunction must satisfy **both** inequalities — both inequalities must be true.

Example 1

Solve and graph the inequality $-1 < 2x + 1 < 5$.

Solution

The aim is to get x by itself.

$$-1 - 1 < 2x + 1 - 1 < 5 - 1 \quad \text{Subtract 1}$$

$$-2 < 2x < 4$$

$$-1 < x < 2$$

Divide by 2 to get x in the middle

So the solution is any number greater than -1 but less than 2 .

This is graphed as: 

✓ Guided Practice

In Exercises 10–13, solve and graph each inequality.

10. $5 < 3y - 1 < 11$

11. $-10 < 4a + 6 < -2$

12. $7 \leq 7x - 7 \leq 14$

13. $-5 \leq 9t + 4 \leq 22$

In Exercises 14–19, solve each inequality.

14. $-5 < \frac{a}{10} < 3$

15. $-11 < \frac{c-9}{7} < 5$

16. $-11 < -4g + 5 < -3$

17. $-9 < 6y - 9 < 3$

18. $-15 < 8y + 9 \leq 9$

19. $7 < \frac{a+6}{-6} \leq 9$

Disjunction Problems Include the Word “Or”

Here’s an example of a disjunction:

$$3x - 4 < -4 \text{ or } 3x - 4 > 4$$

The solution to a disjunction is all the numbers that satisfy **either** one inequality **or** the other.


Example 2

Solve and graph the solution set of $3x - 4 < -4$ or $3x - 4 > 4$.

Solution

$$3x - 4 + 4 < -4 + 4 \quad \text{or} \quad 3x - 4 + 4 > 4 + 4 \quad \text{Add 4}$$

$$3x < 0 \quad \text{or} \quad 3x > 8$$

$$x < 0 \quad \text{or} \quad x > \frac{8}{3} \quad \text{Divide by 3}$$


Check it out:

In this example, the lines head in opposite directions — they have no points in common. Any number in the solution set only satisfies one of the inequalities.

✓ Guided Practice

In Exercises 20–23, solve the inequality and graph each solution set.

20. $7a - 7 < -7$ or $7a - 7 > 21$

21. $5x - 4 \leq 6$ or $5x - 4 \geq 26$

22. $\frac{c-9}{5} < 3$ or $\frac{c-9}{5} \geq 9$

23. $\frac{t-7}{3} \leq -9$ or $\frac{t-7}{3} > 6$

In Exercises 24–27, solve each disjunction.

24. $8c - 4 > 92$ or $8c - 4 < -12$

25. $-9g - 7 \leq 2$ or $-9g - 7 > 20$

26. $6c + 5 > 8$ or $6c + 5 \leq 5$

27. $\frac{j-13}{6} \leq -10$ or $\frac{j-13}{6} \geq 5$

✓ Independent Practice

Solve each conjunction or disjunction in Exercises 1–19.

1. $-7 < 2x + 3 < 9$

2. $8 < 3x - 4 < 14$

3. $9 < 4x + 5 < 17$

4. $-1 \leq x - 3 \leq 5$

5. $3 \leq 3(2x - 5) \leq 9$

6. $-6 \leq 9 - 5x \leq 19$

7. $5 \leq 7 - 2(x - 3) \leq 21$

8. $-2 \leq \frac{3 - 2x}{4} \leq 4$

9. $-3 \leq \frac{5 - 2x}{2} < 4$

10. $3 \leq 4x - 9$ and $4x - 9 \leq 15$

11. $-7 \leq \frac{3(x - 2)}{4}$ and $\frac{3(x - 2)}{4} < 10$

12. $2y + 2 < 4y - 4$ or $4y - 4 > 5y + 2$

13. $5y + 7 < -13$ or $7 - 3y < -5$

14. $11 + 2y < 4y - 3$ or $4y - 3 > 6y + 7$

15. $-2x \leq -3x + 4$ and $-3x + 4 \leq 4x + 18$

16. $\frac{3}{8}x + 3 < -4$ or $3 - \frac{3}{7}x < -6$

17. $-7 \leq \frac{5x - 3}{4} \leq \frac{1}{3}$

18. $-15 \leq 4x - 6 \leq -10$

19. $4x - 9 < 27$ and $10x - 16 > 2x + 8$

20. The sum of three consecutive even integers is between 82 and 85. Find the numbers.

21. The sum of three consecutive odd integers is between 155 and 160. Find the consecutive odd integers.

The formula $C = \frac{5}{9}(F - 32)$ is used to convert degrees Fahrenheit to degrees Celsius. Use this fact to answer Exercises 22–23.

22. The temperature inside a greenhouse falls to a minimum of 65°F at night and rises to a maximum of 120°F during the day. Find the corresponding temperature range in degrees Celsius.

23. The usual temperature range of liquid water is 0 degrees Celsius (freezing point) to 100 degrees Celsius (boiling point). Find the corresponding temperature range of water in degrees Fahrenheit.

Round Up

*This Topic can be a little hard to understand at first. You can write the solution to a **conjunction** in one statement, but **disjunction** solutions usually have to be written in **two parts** because they cover two different parts of the number line.*

Topic 3.4.1

Section 3.4 Absolute Value Inequalities

California Standards:

3.0: Students solve equations and inequalities involving absolute values.

What it means for you:

You'll solve absolute value inequalities of the form $|mx \pm c| < v$ or $|mx \pm c| > v$.

Key words:

- absolute value
- conjunction
- disjunction
- inequality
- endpoint
- interval

You last saw *absolute value equations* in Section 2.8 — now you're going to see *inequalities* involving absolute values.

As with normal inequalities, you can have *conjunctions* and *disjunctions* with absolute value inequalities.

Absolute Value Inequalities are Compound Inequalities

An absolute value inequality can have a form such as $|x| < m$.

$|x| < m$ means that x is **restricted to points** on the number line **less than m units from 0** — in either the positive or the negative direction, as shown on this number line:



$|x| < m$ is equivalent to $-m < x < m$.

✓ Guided Practice

In Exercises 1–4, write the equivalent compound inequality and graph the inequality on a number line.

1. $|x| < 3$

2. $|a| < 7$

3. $|g| < 2$

4. $|x - 1| < 10$

Absolute Value Inequalities Can Be Conjunctions

You can write an **absolute value inequality** of the form $|mx + c| < v$ as the **conjunction** $mx + c < v$ and $mx + c > -v$ (or you could write it $mx + c < v$ and $-(mx + c) < v$).

So, to solve an inequality like this, you can solve the compound inequality $-v < mx + c < v$.

Example 1

Solve $|x - 7| < 11$. Write the solution in interval notation and graph its solution interval on a number line.

Solution

$$x - 7 < 11 \text{ and } x - 7 > -11$$

$$-11 < x - 7 < 11$$

Write the inequality as a conjunction

$$-11 + 7 < x - 7 + 7 < 11 + 7$$

$$-4 < x < 18$$

Solve the conjunction to get x by itself

Solution interval: $(-4, 18)$



Check it out:

The solution interval is the open interval $(-4, 18)$. On the graph, the endpoints -4 and 18 aren't included.

The \leq Sign Means the Endpoints are Included

Example 2

Solve $|4x - 9| \leq 11$. Write the solution in interval notation and graph its solution interval on a number line.

Solution

$$\begin{aligned} -11 &\leq 4x - 9 \leq 11 && \text{Write the inequality as a conjunction} \\ -11 + 9 &\leq 4x - 9 + 9 \leq 11 + 9 \\ -2 &\leq 4x \leq 20 \end{aligned}$$

$$-\frac{1}{2} \leq x \leq 5 \quad \text{Solve the conjunction to get } x \text{ by itself}$$

Solution interval: $[-\frac{1}{2}, 5]$

Graph:



Check it out:

In this example, the solution interval is closed — so the endpoints are included in the graph.

✓ Guided Practice

Solve each conjunction and write each solution set in interval notation.

5. $|t - 10| < 1$

6. $|a + 2| \leq 4$

7. $|3a| \leq 21$

8. $|15j| < 5$

9. $|\frac{r}{8}| \leq 2$

10. $|\frac{h}{12}| < 1$

11. $|x + 8| < 10$

12. $|x - 11| < 4$

13. $|2x - 5| < 11$

14. $|3x - 7| < 8$

15. $|5y + 7| \leq 22$

16. $|\frac{4y + 5}{2}| \leq 13$

17. $|\frac{4y - 1}{3}| \leq 5$

18. $2|\frac{5x - 1}{5}| \leq 4$

19. $|\frac{2x - 3}{2}| \leq 4$

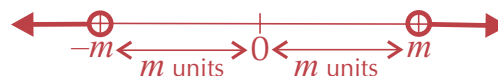
20. $|3(2 - x)| \leq 5$

21. $|5x - 11| \leq 19$

Absolute Value Inequalities Can Be Disjunctions

An absolute value inequality can have a form such as $|x| > m$.

$|x| > m$ means that x is **restricted to points** on the number line **more than m units from 0** — in either the positive or the negative direction, as shown on this number line:



$|x| > m$ is equivalent to $x < -m$ or $x > m$.

Check it out:

The graphs head in opposite directions — they have no points in common.

✓ Guided Practice

In Exercises 22–25, write the equivalent compound inequality, and graph the inequality on a number line.

22. $|x| > 14$

23. $|t| > 8$

24. $|a| > 10$

25. $|t + 8| > 23$

Convert Absolute Values into Normal Disjunctions

An absolute value inequality of the form $|mx + c| > v$ is the same as the disjunction $mx + c < -v$ or $mx + c > v$. To solve this type of absolute value inequality, you solve the disjunction.

Example 3

Solve the compound inequality $|2x - 3| > 7$. Write the solution set in interval notation and graph the solution set.

Solution

$$2x - 3 < -7 \quad \text{or} \quad 2x - 3 > 7 \quad \text{Write the inequality as a disjunction}$$

$$2x < -4 \quad 2x > 10$$

$$x < -2$$

$$x > 5$$

Solve the disjunction to get x by itself

$$\text{Solution set: } (-\infty, -2) \cup (5, \infty)$$

Graph:



Check it out:

In this example, the solution set is in two parts. The endpoints -2 and 5 are not included.

Don't forget:

The “ \cup ” sign is the “union” sign — here it means that the solution set is $(-\infty, -2)$ **and** $(5, \infty)$. See Section 1.1 for a reminder about unions.

The \geq Sign Means the Endpoints are Included

Remember that the solution interval is closed if there is a “greater than or equal to” sign.

Example 4

Solve and graph $|5x + 11| \geq 6$. Write the solution set in interval notation.

Solution

$$5x + 11 \leq -6 \quad \text{or} \quad 5x + 11 \geq 6$$

$$5x \leq -17$$

$$5x \geq -5$$

$$x \leq -\frac{17}{5}$$

$$x \geq -1$$

Solve to get x by itself

$$\text{Solution set: } (-\infty, -3.4] \cup [-1, \infty)$$

Graph:



Check it out:

Here, the solution intervals are half-open. The endpoints -3.4 and -1 are included.

✓ Guided Practice

Solve each disjunction and write each solution set in interval notation.

26. $|a - 8| > 1$

27. $|t + 2| > 8$

28. $|7a| \geq 14$

29. $|4j| \geq 16$

30. $\left|\frac{c}{3}\right| > 6$

31. $\left|\frac{c}{12}\right| \geq \frac{1}{3}$

32. $|x + 9| > 7$

33. $|x - 11| > 12$

34. $|3x - 7| > 13$

35. $|5y + 11| > 21$

36. $|4m + 9| \geq 11$

37. $|7b - 8| \geq 13$

38. $\left|\frac{4x - 3}{3}\right| \geq 5$

39. $\left|\frac{2x - 1}{7}\right| \geq 3$

40. $|3(x - 2) + 7| \geq 8$

41. $|4(3 + x)| \geq 13$

42. $|2(3x - 7) + 15| \geq 11$

✓ Independent Practice

1. Write $|a| > 5$ as a compound inequality.

2. Write $|w| \leq c$ as a compound inequality.

In Exercises 3–4 write the equivalent compound inequality in interval notation.

3. $|a| < 4$

4. $|b| \geq 2$

In Exercises 5–12, solve each inequality.

5. $\left|\frac{g}{2}\right| \geq 3$

6. $|c + 2| > 24$

7. $|w - 17| \leq 8$

8. $|4a - 3| \leq 9$

9. $|5t + 10| \geq 15$

10. $\left|\frac{3x + 1}{2}\right| < 1$

11. $\left|\frac{7t - 8}{3}\right| \geq 2$

12. $\frac{2|c + 17|}{5} > 4$

13. Solve and graph the solution set of $7|2(x - 7) + 5| \geq 14$.

14. Given $m > 0$, graph the solution set of $|x| \leq m$ on a number line.

15. Given $m > 0$, solve and graph the solution set of $|x| \geq m$.

16. Solve $|3m - 5| > m + 7$.

17. Solve $|5m + 3| > 2m - 1$.

18. A floor tile must measure 20 cm along its length, to within 2 mm. Write and solve an absolute value inequality to find the maximum and minimum possible lengths for the tile.

Round Up

Like in Section 3.3, you've got to watch out for the difference between *conjunctions* and *disjunctions*. Also, look back at Section 1.1 if you've forgotten what the " \cup " sign means.

Chapter 3 Investigation

Mailing Packages

Inequalities turn up all the time in everyday life. In this investigation you'll see that using inequalities makes it much easier to model some real-life problems.

A shipping company will deliver packages of any weight, as long as the conditions on the right are satisfied.

Part 1:

Design a set of six boxes of different shapes and sizes that meet these requirements.

Part 2:

What is the **maximum volume** a box can have while still satisfying the shipping company's requirements?

Packages must satisfy the following:

$$\text{Length} + \text{Width} + \text{Height} \leq 120 \text{ cm}$$

$$\text{Length} \leq 80 \text{ cm}$$

$$\text{Width} \leq 80 \text{ cm}$$

$$\text{Height} \leq 80 \text{ cm}$$

$$\text{Volume} = \text{length} \times \text{width} \times \text{height}$$

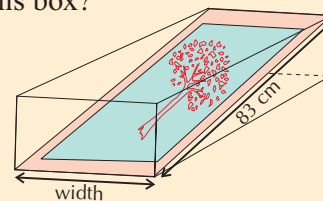
Things to think about:

Try to include a wide range of **different sizes and shapes** in your set of boxes. Aim to make your set as **useful as possible** for mailing different types of item.

Extension

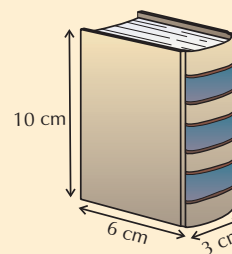
For these extensions, only boxes in the shapes of **rectangular prisms** should be considered.

- 1) What is the **maximum surface area** a box can have while still satisfying the shipping company's requirements? What are the **dimensions** of this box?
- 2) You need to ship a picture that has a height of **83 cm**. How wide could the picture be so that it fits in a rectangular prism shaped box that satisfies the shipping company's requirements? Write an **inequality** to represent the possible width of the picture.



Open-ended Extension

- 1) You work for an organization that produces pocket dictionaries. The dimensions of the dictionaries are shown on the right. What size box will hold the **greatest** number of dictionaries, while still satisfying the shipping company's requirements? The dictionaries **don't** all have to be placed in the box the same way. Experiment with placing some dictionaries vertically and some horizontally.
- 2) Your company is concerned about the environment and wants to use the **smallest area of cardboard** possible to package the books in. Design the box that you think will be most efficient. (Assume the box does not need any tabs to stick it together.)



Round Up

In general, problems that use the words "maximum," "minimum," "limits," or "tolerance" often mean that you need to use inequalities to model the situations.

Chapter 4

Linear Equations and Their Graphs

Section 4.1	The Coordinate Plane	164
Section 4.2	Lines	175
Section 4.3	Slope	189
Section 4.4	More Lines	197
Section 4.5	Inequalities	212
Investigation	Tree Growth	227

Topic 4.1.1

Section 4.1 The Coordinate Plane

California Standards:

6.0: Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

What it means for you:

You'll refresh your knowledge of plotting coordinates.

Key words:

- coordinate
- horizontal
- vertical
- point of intersection
- origin

Plotting coordinates isn't anything new to you — you've had lots of practice in earlier grades. This Topic starts right at the beginning though, to remind you of the earlier work.

Coordinates are Used to Locate Points on a Plane

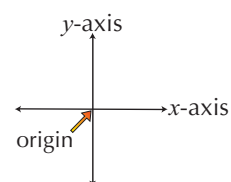
A plane is a **flat surface**, kind of like a blackboard, a tabletop, or a sheet of paper. However, a plane extends **indefinitely** in **all directions** — it goes on and on forever.

Planes are made up of an **infinite number of points**. To locate one of these points, **two perpendicular number lines** are drawn in the plane.

The **horizontal** number line is called the **x -axis**.

The **vertical** number line is called the **y -axis**.

The **point of intersection** of the two number lines is called the **origin**.



You can locate each point in the plane using an **ordered pair** of numbers (x, y) , where **x** represents the **horizontal** distance and **y** represents the **vertical** distance of the point from the origin.

The numbers in an ordered pair are often called **coordinates**.

Check it out:

The x -coordinate is sometimes called the abscissa, and the y -coordinate is sometimes called the ordinate.

Check it out:

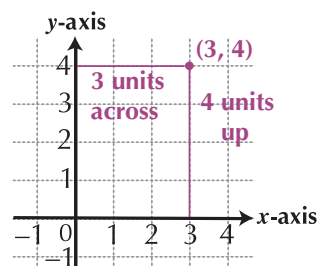
The coordinates of the origin are $(0, 0)$.

Example 1

Plot the coordinates $(3, 4)$ on a graph.

Solution

$(3, 4)$
↑ x -coordinate ↑ y -coordinate



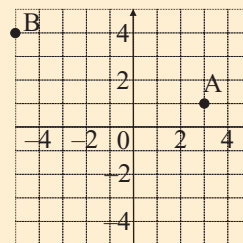
✓ Guided Practice

In Exercises 1–4, plot the coordinates on a coordinate plane.

1. $(4, 0)$
2. $(-2, 5)$
3. $(8, 2)$
4. $(5, -2)$

Use the graph opposite to answer Exercises 5–6.

5. What are the coordinates of point A?
6. What are the coordinates of point B?



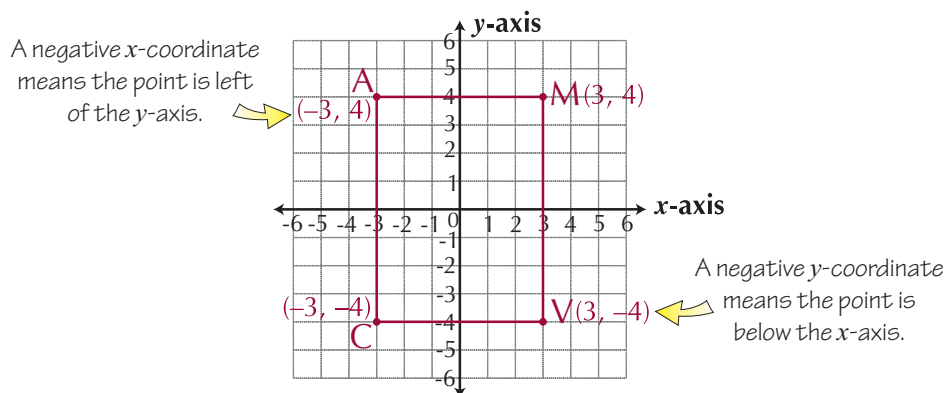
You May Need to Plot More Than One Point

You can often join up several plotted points to create the outline of a shape.

Example 2

Draw a coordinate plane and plot and label the points M $(3, 4)$, A $(-3, 4)$, C $(-3, -4)$, and V $(3, -4)$. Connect each pair of consecutive points and find the perimeter of the resulting quadrilateral.

Solution



The length of each square on the coordinate grid represents one unit of measure. So, the quadrilateral is 8 units long and 6 units wide.

$$\begin{aligned}\text{Perimeter (distance around the edge)} &= 2(6) + 2(8) \\ &= 12 + 16 \\ &= \mathbf{28 \text{ units}}\end{aligned}$$

✓ Guided Practice

Work out Exercises 7–9 by plotting and labeling the points on copies of the coordinate plane used in Example 2.

7. A (0, 5), B (–5, 2), C (–5, –4), D (5, –4), and E (5, 2).

Name the figure formed when the points are connected in order.

8. A (3, 3), B (–3, 3), C (–3, –3), and D (3, –3).

Connect the points in order and then name and find the area of the figure formed.

9. A (2, 5), B (2, –2), C (5, –2), D (5, –4), E (–5, –4), F (–5, –2), G (–2, –2), and H (–2, 5).

Connect the points in order and find the perimeter of the figure formed.

✓ Independent Practice

1. What is the difference between the x -axis and the y -axis?

2. Explain in words how to graph the coordinates (5, 3).

Use the graph opposite to answer Exercises 3–7.

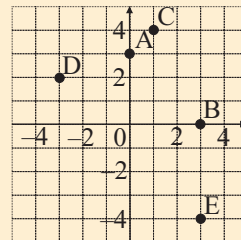
3. What are the coordinates of point A?

4. What are the coordinates of point B?

5. What are the coordinates of point C?

6. What are the coordinates of point D?

7. What are the coordinates of point E?



In Exercises 8–11, plot the coordinates on a coordinate plane.

8. The origin

9. (2, 1)

10. (–4, 1)

11. (3, –2)

12. Plot and label the following points:

A (–1, –2),

B (2, 4),

C (5, –2),

D (–2, 2),

and E (6, 2).

Connect the points in order, then connect E to A, and name the figure formed.

Round Up

*That Topic should have felt quite familiar. Remember that coordinate pairs always list the x -coordinate first, then the y -coordinate — and watch out for any **negative numbers**. In the next Topic you'll look at each part of the coordinate plane in more detail.*

Quadrants of the Plane

California Standards:

6.0: Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

What it means for you:

You'll look in more detail at the four main regions of the coordinate plane, and the axes.

Key words:

- coordinate
- quadrant
- axes

There are *four main regions* of the coordinate plane — they're *divided up* by the x - and y -axes.

This Topic is about spotting where on the coordinate plane points lie.

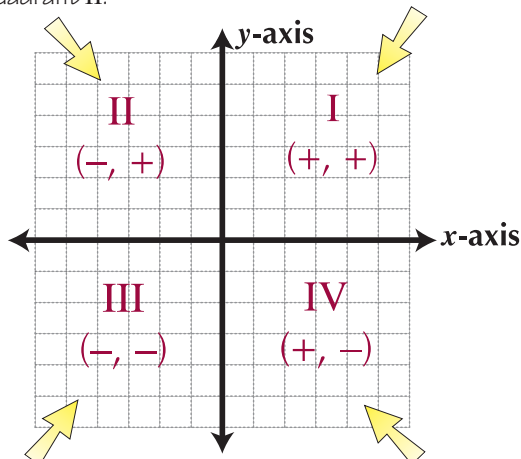
The Plane Consists of Four Quadrants

The coordinate axes divide the plane into four regions called **quadrants**. The quadrants are numbered **counterclockwise** using **Roman numerals**.

The **signs of the coordinates** differ from quadrant to quadrant, as shown in the diagram below.

If the x -coordinate is negative and the y -coordinate is positive, the point is in quadrant II.

If the x - and y -coordinates are both positive, the point is in quadrant I.



If the x - and y -coordinates are both negative, the point is in quadrant III.

If the x -coordinate is positive and the y -coordinate is negative, the point is in quadrant IV.

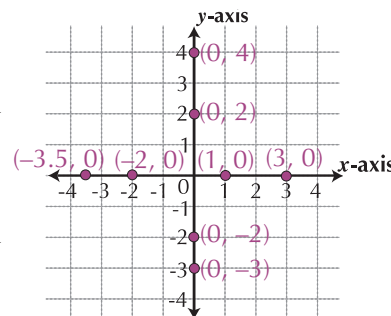
So, you can easily tell **which quadrant** a particular point is in by simply looking at the signs of its coordinates.

Some Points are on the x - or y -Axes

Points in the plane are located in one of the four quadrants or **on the axes**.

All points whose coordinates are in the form $(x, 0)$ are on the **x -axis** — for example, $(3, 0)$, $(1, 0)$, $(-2, 0)$, $(-3.5, 0)$.

All points whose coordinates are in the form $(0, y)$ are on the **y -axis** — for example, $(0, 2)$, $(0, 4)$, $(0, -3)$, $(0, -2)$.



Example 2

State which axis, if either, these points lie on. Justify your answer.

- a) $(0, 3)$ b) $(12, 2)$ c) $(45, 0)$

Solution

- a) $(0, 3)$ is on the **y -axis**, since $x = 0$.
b) $(12, 2)$ **isn't on an axis**, since neither x nor y is 0.
c) $(45, 0)$ is on the **x -axis**, since $y = 0$.

✓ Guided Practice

In Exercises 19–24, let $t > 0$. State which axis, if either, each point lies on.

19. $(0, 1)$ 20. $(0, -3)$ 21. $(t, 1)$
22. $(t, 0)$ 23. $(-4, 0)$ 24. $(t + 1, 0)$

✓ Independent Practice

In Exercises 1–12, let $a > 0$ and $b < 0$. Name the quadrant or axis where each point is located, and justify your answers.

1. $(5, a)$ 2. $(a, 4)$ 3. $(a + 2, 4)$
4. $(0, -3)$ 5. $(-2, 0)$ 6. $(0, 0)$
7. $(b, 1)$ 8. $(b, 3)$ 9. $(2a, b)$
10. $(2b, a)$ 11. $(-b, a)$ 12. $(-a, -2b)$

Round Up

Another straightforward Topic, really. Working out which **quadrant** a coordinate pair appears in is all about **checking the signs** of each of the numbers — and a coordinate pair can only be on one of the axes if one of the numbers is 0. In the next Topic you'll look at lines plotted on the coordinate plane.

Lines

California Standards:

6.0: Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

What it means for you:

You'll learn a formal definition of a line, and you'll refresh your knowledge of line equations.

Key words:

- coordinate
- quadrant
- axes

*That's enough of learning about the coordinate plane itself — now it's time to **plot lines**. If you **join two points** in a coordinate plane you'll form part of a **line**.*

Lines Have No Endpoints

A straight line extends **indefinitely** in opposite directions.

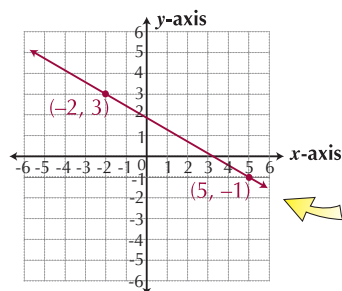
- Lines have:
- **no endpoints** (they have no beginning or end)
 - **infinite length**
 - **no thickness**

To identify a straight line you just need **two points** on it. So, to draw a straight line in a plane, simply **plot two points** and connect them using a **straightedge**.

Example 1

Draw the straight line defined by the points $(-2, 3)$ and $(5, -1)$.

Solution



The arrowheads show that the line continues indefinitely (without ending).

Guided Practice

Complete the sentence in Exercise 1.

1. A straight line extends _____ in opposite directions.

In Exercises 2–8, draw the straight line defined by the pairs of coordinates.

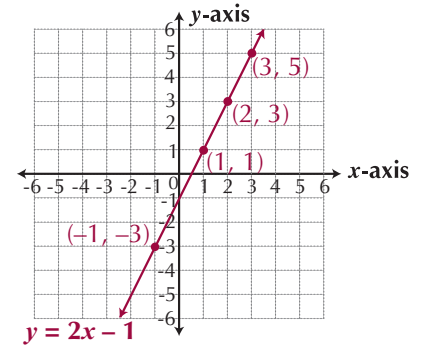
2. $(3, 1)$ and $(-4, 2)$
3. $(0, 0)$ and $(-3, -3)$
4. $(-2, 1)$ and $(3, 1)$
5. $(3, -2)$ and $(-1, -2)$
6. $(1, 3)$ and $(1, -4)$
7. $(5, 1)$ and $(-3, 2)$
8. $(3, 3)$ and $(-1, -1)$

A Line Can Be Described by an Equation

Check it out:

At the point $(3, 5)$, $x = 3$.
 You can check that $(3, 5)$ is of the form $(x, 2x - 1)$ by substituting 3 for x .
 In other words:
 $(x, 2x - 1) \Rightarrow (3, 2(3) - 1)$
 $\Rightarrow (3, 5)$.

All the points that lie on the line shown have coordinates (x, y) , where $y = 2x - 1$. This means that the coordinates all have the form $(x, 2x - 1)$ — for example, $(-1, -3)$, $(1, 1)$, $(2, 3)$, $(3, 5)$.



$y = 2x - 1$ is called the **equation** of the line.

Example 2

The points on the line $y = 7x + 3$ are defined by $(x, 7x + 3)$. Find the coordinates of the points where $x \in \{0, 1, 2\}$.

Solution

Just substitute each value of x into $(x, 7x + 3)$ to find the coordinates.

$$x = 0 \text{ means } (x, 7x + 3) = (0, 7(0) + 3) = (0, 3)$$

$$x = 1 \text{ means } (x, 7x + 3) = (1, 7(1) + 3) = (1, 10)$$

$$x = 2 \text{ means } (x, 7x + 3) = (2, 7(2) + 3) = (2, 17)$$

So the coordinates of the points are $(0, 3)$, $(1, 10)$, and $(2, 17)$.

Guided Practice

Draw out a coordinate grid spanning -6 to 6 on the x -axis and -6 to 6 on the y -axis. Draw and label the following lines on the grid.

9. $y = 3$

10. $y = -4$

11. $y = 0$

12. $x = 2$

13. $x = -5$

14. $x = 0$

Draw the graphs for Exercises 15–16 on coordinate grids spanning -6 to 6 on the x -axis and -6 to 6 on the y -axis.

15. Draw the graph of the set of all points (x, y) such that $x = y$.

16. Draw the graph of the set of points (x, y) such that $x = 5$ and $y \in \mathbb{R}$. Describe the line you have drawn.

17. If $x \in \{-1, 1, 3\}$, find the set M of points defined by $(x, -2x + 1)$.

Exercises 18 and 19 are about the set of ordered pairs $(x, 5x - 6)$.

18. Name the five members of the set if x is a natural number less than 6.

19. Which members of this set are also members of the set of ordered pairs (x, x^2) ?

✓ Independent Practice

In Exercises 1–2, x is an integer greater than -4 **and** less than 0 .

1. Find the set of points defined by $(x, 3x - 2)$.
2. Which member of this set is also a member of the set of ordered pairs $(x, x - 4)$?

Work out Exercises 3–4 by plotting and labeling the points on a grid spanning -6 to 6 on the x -axis and -3 to 9 on the y -axis.

3. If $x \in \{-3, -2, -1, 0, 1, 2, 3\}$, plot the set of points defined by (x, x^2) .
4. If $x \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$, plot the set of points defined by $(x, |x + 1|)$.
5. What shape would you expect the graph to be for the set of points defined by $(x, |x|)$?

Work out Exercises 6–11 by plotting and labeling the points on a grid spanning -6 to 6 on the x -axis and -6 to 6 on the y -axis.

6. Plot two lines l_1 and l_2 whose points are defined by $l_1 = (x, 2x + 1)$ and $l_2 = (x, 2x + 3)$.
7. Describe the two lines defined in Exercise 6 and the relationship between them.
8. Plot the two lines whose points are defined by $(x, -2x + 1)$ and $(x, -2x - 2)$.
9. Describe the two lines defined in Exercise 8 and the relationship between them.
10. Draw two lines whose points are defined by $(x, 2x + 1)$ and $(x, -0.5x + 1)$.
11. Give the coordinates of the point where the two lines defined in Exercise 10 intersect.

Work out Exercises 12–14 by plotting and labelling the points on a grid spanning -6 to 6 on the x -axis and -9 to 3 on the y -axis.

12. If $x \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$, plot all points defined by $(x, -|x + 1|)$.
13. Compare the graph of the set of points defined by $(x, |x + 1|)$ (in Exercise 4) with the graph of the set of points defined by $(x, -|x + 1|)$ above. Describe the relationship between the two graphs.
14. What would you expect the relationship to be between the graph of the set of points defined by $(x, |x|)$ and the graph of the set of points defined by $(x, -|x|)$?

Round Up

*You know what a line is from everyday life — but notice that the Math definition is a little more precise. In Math, a line is **infinitely long** and **doesn't actually have any thickness**. It's hard to imagine, but it's OK to just carry on without worrying too much about how strange that seems.*

Topic 4.1.4

California Standards:

6.0: Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

What it means for you:

You'll check whether lines are horizontal or vertical.

Key words:

- horizontal
- vertical
- constant

Horizontal and Vertical Lines

You can tell a lot about a line just by looking at the *points it goes through*. One of the simplest things to spot without plotting the graph is whether the line is *horizontal* or *vertical*.

Points on a Vertical Line Have the Same x -Coordinate

The **x -coordinate** tells you how far to the **left or right** of the y -axis a point is. Points with the **same x -coordinate** are all the **same horizontal distance** from the y -axis.

So, if a set of points all have the **same x -coordinate**, that set will fall on a **vertical line**.

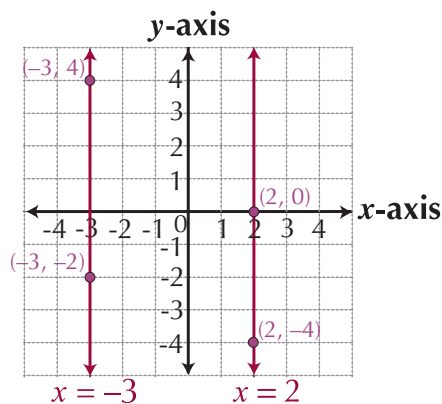
The equation of a **vertical line** is $x = c$, where c is a constant (fixed) number.

For example, $x = 3$, $x = -1$.

Example 1

Draw and label the lines $x = -3$ and $x = 2$.

Solution



The values of the y -coordinates are different for each point on the line, but the values of x are the same (-3 for the line $x = -3$, 2 for the line $x = 2$).

Points on a Horizontal Line have the Same y -Coordinate

The **y -coordinate** tells you how far **above or below** the x -axis a point is. Points with the **same y -coordinate** are all the **same vertical distance** from the x -axis.

So, if a set of points all have the **same y -coordinate**, that set will fall on a **horizontal line**.

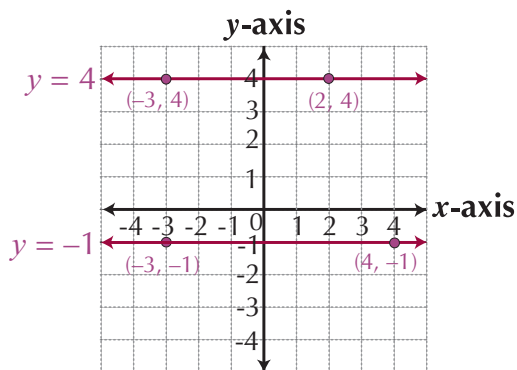
The equation of a **horizontal line** is $y = c$, where c is a constant (fixed) number.

For example, $y = 6$, $y = -3$.

Example 2

Draw and label the lines $y = 4$ and $y = -1$.

Solution



The values of the x -coordinates are different for each point on the line, but the values of y are the same (-1 for the line $y = -1$, 4 for the line $y = 4$).

Guided Practice

In Exercises 1–10, draw and label each set of lines on a coordinate plane.

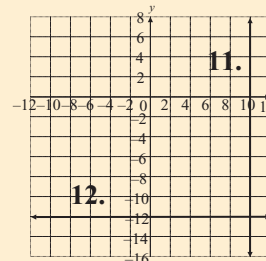
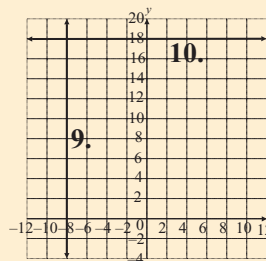
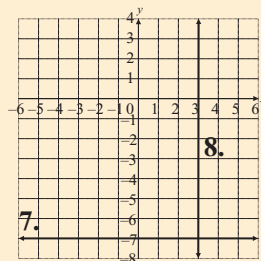
- | | |
|--------------------------|-------------------------|
| 1. $x = 3$ and $x = -1$ | 2. $x = 4$ and $x = 1$ |
| 3. $x = 2$ and $x = -3$ | 4. $y = 3$ and $y = -1$ |
| 5. $y = 4$ and $y = 1$ | 6. $y = 2$ and $y = -3$ |
| 7. $y = 3$ and $x = 2$ | 8. $y = 6$ and $x = -4$ |
| 9. $y = -7$ and $x = -5$ | 10. $x = 6$ and $y = 4$ |

Independent Practice

In Exercises 1–6, draw and label each set of lines on a coordinate plane.

- | | |
|--------------------------|-------------------------|
| 1. $x = 8$ and $x = 0$ | 2. $x = -3$ and $x = 1$ |
| 3. $x = 4$ and $x = -6$ | 4. $y = 0$ and $y = 3$ |
| 5. $y = -4$ and $y = -6$ | 6. $y = 2$ and $y = -3$ |

In Exercises 7–12, write the equation for each line on the graph.



Round Up

One thing that can look confusing at first is that the line $x = 0$ is actually the y -axis, while the line $y = 0$ is the x -axis. And a line of the form $x = c$ (for any constant c) is a **vertical line** (that is, it is parallel to the y -axis) — while a line $y = c$ is a **horizontal line** (that is, it's parallel to the x -axis).

Topic 4.2.1

Section 4.2 Points on a Line

California Standards:

6.0: Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

7.0: Students verify that a point lies on a line, given an equation of the line.

Students are able to derive linear equations by using the point-slope formula.

What it means for you:

You'll learn how to show mathematically that points lie on a line.

Key words:

- linear equation
- variable
- solution set
- verify

You already dealt with lines in Topics 4.1.3 and 4.1.4. In this Topic you'll see a *formal definition* relating *ordered pairs* to a *line* — and you'll also learn how to *show* that *points lie on a particular line*.

Graphs of Linear Equations are Straight Lines

An equation is **linear** if the variables have an exponent of one and there are no variables multiplied together.

For example,

linear: $3x + y = 4$, $2x = 6$, $y = 5 - x$

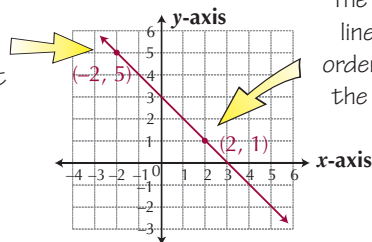
nonlinear: $xy = 12$, $x^2 + 3y = 1$, $8y^3 = 20$

Linear equations in **two variables**, x and y , can be written in the form **$Ax + By = C$** .

The **solution set** to the equation $Ax + By = C$ consists of all ordered pairs (x, y) that satisfy the equation. All the points in this solution set lie on a **straight line**. This straight line is the **graph** of the equation.

If the ordered pair (x, y) satisfies the equation $Ax + By = C$, then the point (x, y) lies on the graph of the equation.

The ordered pair $(-2, 5)$ satisfies the equation $2x + 2y = 6$. So the point $(-2, 5)$ lies on the line $2x + 2y = 6$.



The point $(2, 1)$ lies on the line $2x + 2y = 6$. So the ordered pair $(2, 1)$ satisfies the equation $2x + 2y = 6$.

Verifying That Points Lie on a Line

To determine whether a point (x, y) **lies on the line** of a given equation, you need to find out whether the ordered pair (x, y) **satisfies the equation**. If it does, the point is on the line. You do this by **substituting** x and y into the equation.

Example 1**Check it out:**

This is the method for showing that $(2, -3)$ is a solution of $x - 3y = 11$.

- a) Show that the point $(2, -3)$ lies on the graph of $x - 3y = 11$.
 b) Determine whether the point $(-1, 1)$ lies on the graph of $2x + 3y = 4$.

Solution

a) $2 - 3(-3) = 11$ **Substitute 2 for x and -3 for y**

$$2 + 9 = 11$$

$$11 = 11$$
 A true statement

So the point $(2, -3)$ lies on the graph of $x - 3y = 11$, since $(2, -3)$ satisfies the equation $x - 3y = 11$.

b) If $(-1, 1)$ lies on the line, $2(-1) + 3(1) = 4$.

$$\text{But } 2(-1) + 3(1) = -2 + 3$$

$$= 1$$

Since $1 \neq 4$, $(-1, 1)$ does not lie on the graph of $2x + 3y = 4$.

✓ Guided Practice

Determine whether or not each point lies on the line of the given equation.

1. $(-1, 2)$; $2x - y = -4$

2. $(3, -4)$; $-2x - 3y = 6$

3. $(-3, -1)$; $-5x + 3y = 11$

4. $(-7, -3)$; $2y - 3x = 15$

5. $(-2, -2)$; $y = 3x + 4$

6. $(-5, -3)$; $-y + 2x = -7$

7. $(-2, -1)$; $8x - 15y = 3$

8. $(1, 4)$; $4y - 12x = 3$

9. $\left(\frac{1}{3}, -\frac{1}{4}\right)$; $6x - 16y = 7$

10. $\left(\frac{2}{3}, -\frac{2}{5}\right)$; $-3x - 10y = 2$

✓ Independent Practice

In Exercises 1–4, determine whether or not each point lies on the graph of $5x - 4y = 20$.

1. $(0, 4)$

2. $(4, 0)$

3. $(2, -3)$

4. $(8, 5)$

In Exercises 5–8, determine whether or not each point lies on the graph of $6x + 3y = 15$.

5. $(2, 1)$

6. $(0, 5)$

7. $(-1, 6)$

8. $(3, -1)$

In Exercises 9–12, determine whether or not each point lies on the graph of $6x - 6y = 24$.

9. $(4, 0)$

10. $(1, -3)$

11. $(100, 96)$

12. $(-400, -404)$

13. Explain in words why $(2, 31)$ is a point on the line $x = 2$ but not a point on the line $y = 2$.

14. Determine whether the point $(3, 4)$ lies on the lines $4x + 6y = 36$ and $8x - 7y = 30$.

Round Up

You can always *substitute x and y into the equation to prove whether a coordinate pair lies on a line.* That's because if the coordinate pair lies on the line then it's actually a *solution* of the equation.

Graphing $Ax + By = C$

California Standards:

6.0: Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

7.0: Students verify that a point lies on a line, given an equation of the line.

Students are able to derive linear equations by using the point-slope formula.

What it means for you:

You'll learn how to graph a straight line by joining two points.

Key words

- linear equation

Every point on a line is a solution to the equation of the line.

If you know any two solutions (any two coordinate pairs), then you can join the points with a straight line.

Graphing the Line $Ax + By = C$ Using Two Points

The **graph** of the equation $Ax + By = C$ consists of all points (x, y) whose coordinates **satisfy** $Ax + By = C$. To **graph** the line, you just need to plot **two** points on it and join them together with a straight line.

- **Rearrange** the equation so it is in the form $y = Px + Q$.
- Choose **two values of x** and **substitute** them into your equation to find the corresponding values of y .
- **Plot** the two points and draw a **straight line** through them.
- Plot a **third** point to check that the line is correct — the point should lie on the line.

Example 1

Plot and label the graph of the equation $x - y = -3$.

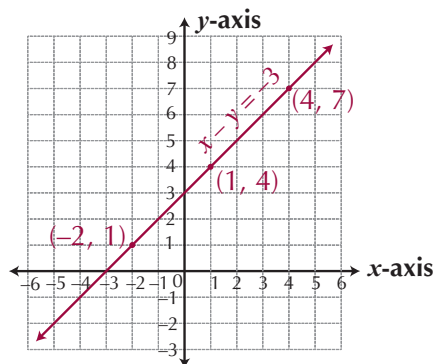
Solution

Rearrange this first to get $y = x + 3$.

Choose two values of x , then draw a **table** to help you **find the y -values**.

x	y	(x, y)
-2	$y = x + 3 = -2 + 3 = 1$	$(-2, 1)$
4	$y = x + 3 = 4 + 3 = 7$	$(4, 7)$

When you plot the graph, the line should be **straight**.



Check:

$$x = 1$$

$$\Rightarrow y = x + 3 = 1 + 3 = 4$$

$$\Rightarrow (x, y) = (1, 4)$$

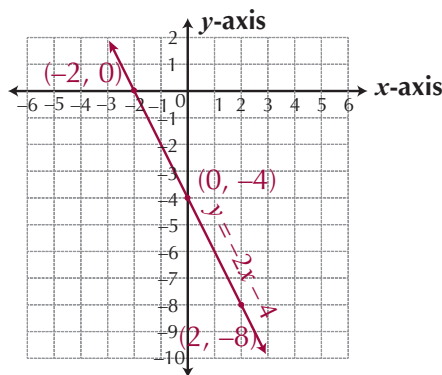
$(1, 4)$ lies on the line — which means **the line is correct**.

Example 2

Graph and label the line of the equation $y = -2x - 4$.

Solution

x	y	(x, y)
-2	$y = -2x - 4 = -2(-2) - 4 = 0$	$(-2, 0)$
2	$y = -2x - 4 = -2(2) - 4 = -8$	$(2, -8)$



Check:

$$x = 0$$

$$\Rightarrow y = -2x - 4 = -2(0) - 4 = 0 - 4 = -4$$

$$\Rightarrow (x, y) = (0, -4)$$

$(0, -4)$ lies on the line — which means that **the line is correct**.

✓ Guided Practice

Graph the line through the two points in each of Exercises 1–2.

1. $(-1, -3)$ and $(3, 5)$

2. $(-3, 4)$ and $(4, -3)$

Graph and label the lines of the equations in Exercises 3–6.

3. $-x - 2y = 4$

4. $2x - 3y = 6$

5. $5y - 3x = 15$

6. $7y - 2x = -14$

✓ Independent Practice

In Exercises 1–4, graph the line through each set of two points.

1. $(-1, -2)$ and $(2, 4)$

2. $(-1, -1)$ and $(1, 3)$

3. $(0, 0)$ and $(2, 6)$

4. $(0, -2)$ and $(1, 1)$

Graph and label the lines of the equations in Exercises 5–16.

5. $x + y = 8$

6. $y - x = 10$

7. $2x + y = -3$

8. $5x + y = -12$

9. $-3x + y = -6$

10. $-10x + y = 21$

11. $2x - y = -14$

12. $6x + 2y = 18$

13. $8x + 4y = 24$

14. $12x - 4y = 8$

15. $3x - 9y = -27$

16. $2x - 8y = 16$

Round Up

It's easy to make a mistake when working out y -values, so choose x -values that will make the algebra easy (for example, 0 and 1). And it's always a good idea to check your line by plotting a third point.

Using Line Equations

California Standards:

7.0: Students verify that a point lies on a line, given an equation of the line.

Students are able to derive linear equations by using the point-slope formula.

What it means for you:

You'll rearrange line equations to find unknown values.

Key words:

- linear equation
- substitute

Check it out:

Because $(2k, 3)$ lies on the line you can replace x and y with $2k$ and 3 , and the equation will still hold.

This Topic carries straight on from Topic 4.2.2.

*If there's an unknown in a pair of coordinates then you need to **substitute** the x and y values into the equation, and **rearrange** to find the unknown value.*

Points on a Line Satisfy the Equation of the Line

In Lesson 4.2.1 you saw that if the ordered pair (x, y) satisfies the equation $Ax + By = C$, then the point (x, y) lies on the corresponding graph.

Similarly, **if a point (x, y) lies on the line $Ax + By = C$, then its coordinates satisfy the equation.**

So if you **substitute** the coordinates for x and y into the equation, **it will make a true statement.** Here are some examples showing how you can use this property to calculate unknown values:

Example 1

$(2k, 3)$ is a point on the line $x - 3y = 7$. Find k . Justify your answer.

Solution

$$x - 3y = 7$$

$$2k - 3(3) = 7$$

Substituting $2k$ for x and 3 for y

$$2k - 9 = 7$$

$$2k - 9 + 9 = 7 + 9$$

Addition Property of Equality

$$2k = 16$$

$$\frac{2k}{2} = \frac{16}{2}$$

Division Property of Equality

$$k = 8$$

So the point on the line $x - 3y = 7$ with y -coordinate 3 is $(16, 3)$.

Example 2

$\left(4, \frac{2}{3}m\right)$ is a point on the line $2x - 9y = -10$. Find the value of m .

Solution

$$2x - 9y = -10$$

$$2(4) - 9\left(\frac{2}{3}m\right) = -10$$

$$8 - 6m = -10$$

$$-6m = -18$$

$$\frac{-6m}{-6} = \frac{-18}{-6}$$

$$m = 3$$

So the point on the line $2x - 9y = -10$ with x -coordinate 4 is $(4, 2)$.

 **Guided Practice**

1. $\left(\frac{2}{3}p, 4\right)$ lies on the graph of $6x - 5y = -32$. Find the value of p .

2. The point $\left(-2, \frac{4}{5}h\right)$ lies on the graph of $-x - 3y = 3$.

Find the value of h .

3. The point $(2a, -3a)$ lies on the graph of $-x - 2y = -12$.

Find the value of a .

4. Find the coordinates of the point in Exercise 5.

5. $\left(\frac{1}{4}k, -\frac{3}{4}k\right)$ is a point on the line $x - 3y = 5$. Find the value of k .

6. Find the coordinates of the point in Exercise 7.

Example 3

The point $(1, 3)$ lies on the line $bx + y = 6$. Find the value of b .

Solution

Here you have to use the coordinates to identify the equation. The question is different, but the method is the same.

$$\begin{aligned} bx + y &= 6 \\ b(1) + 3 &= 6 \\ b + 3 &= 6 \\ \mathbf{b} &= \mathbf{3} \end{aligned}$$

✓ Guided Practice

7. The point $(-1, 2)$ lies on the line $-bx + 2y = -4$. Find the value of b .
8. The point $(-3, -5)$ lies on the graph of $2x - 3ky = 24$. Find the value of k .
9. The point $(8, 7)$ lies on the line $bx + y = 11$. Find b .
10. The point $(6, 3)$ lies on the line $bx - y = 9$. Find b .
11. The point $(14, 3)$ lies on the line $x + by = -10$. Find b .
12. The point $(23, -4)$ lies on the line $x - ky = -21$. Find k .
13. The point $(4, 4)$ lies on the line $4x - 2ky = 0$. Find k .
14. The point $(5, -1)$ lies on the line $9x + 3ky = 30$. Find k .
15. The point $(-2, -2)$ lies on the line $7kx - 3y = 13$. Find k .
16. The point $(-1, -4)$ lies on the line $7kx - 2ky = 0.25$. Find k .

✓ Independent Practice

In Exercises 1–5, the given point lies on the line $3x - 4y = 24$. Find the value of each k and find the coordinates of each point.

1. $(2k, 0)$
2. $(4k, -6)$
3. $(4k, -18)$
4. $(4k, k)$
5. $(-4k, -6k)$

In Exercises 6–10, the point $(2, -4)$ lies on the given line. Find the value of b in each case.

6. $bx + 3y = -6$
7. $-bx + 5y = 10$
8. $6x - by = 32$
9. $7x + by = -18$
10. $9bx - 2by = 4$

11. The point $(1, 1)$ lies on the line $bx - 2by = 4$. Find b .
12. The point $(-3, 6)$ lies on the line $4x + 6ky = 24$. Find k .
13. The point $(4k, 2k)$ lies on the line $2x - 6y = 12$. Find k and the coordinates of the point.

Round Up

This Topic's really just an application of the method you learned in Topic 4.2.1.

Once you've substituted the x and y values into the equation then you just solve it as normal.

Topic 4.2.4

California Standards:

6.0: Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

What it means for you:

You'll learn about x - and y -intercepts and how to compute them from the equation of a line.

Key words:

- intercept
- linear equation

The x - and y -Intercepts

The *intercepts* of a graph are the points where the graph crosses the *axes*. This Topic is all about how to calculate them.

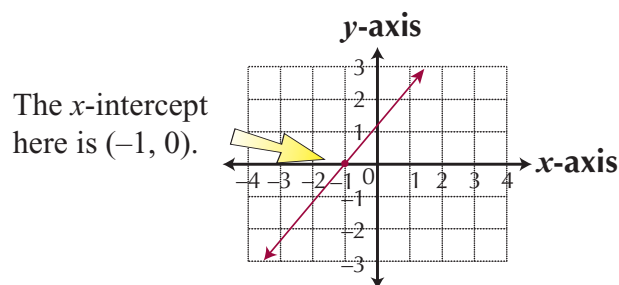
The x -Intercept is Where the Graph Crosses the x -Axis

The x -axis on a graph is the horizontal line through the origin.

Every point on it has a y -coordinate of 0.

That means that all points on the x -axis are of the form $(x, 0)$.

The x -intercept of the graph of $Ax + By = C$ is the **point** at which the graph of $Ax + By = C$ **crosses the x -axis**.



Computing the x -Intercept Using “ $y = 0$ ”

Since you know that the x -intercept has a y -coordinate of 0, you can find the x -coordinate by letting $y = 0$ in the equation of the line.

Example 1

Find the x -intercept of the line $3x - 4y = 18$.

Solution

Let $y = 0$, then solve for x :

$$3x - 4y = 18$$

$$3x - 4(0) = 18$$

$$3x - 0 = 18$$

$$3x = 18$$

$$x = 6$$

So $(6, 0)$ is the x -intercept of $3x - 4y = 18$.

Example 2

Find the x -intercept of the line $2x + y = 6$.

Solution

Let $y = 0$, then solve for x :

$$2x + y = 6$$

$$2x + 0 = 6$$

$$2x = 6$$

$$x = 3$$

So **(3, 0)** is the x -intercept of $2x + y = 6$.

Check it out:

Always write the x -intercept as a point, not just as the value of x where the graph crosses the x -axis. For example, (6, 0), not 6.

✓ Guided Practice

In Exercises 1–8, find the x -intercept.

1. $x + y = 5$

2. $3x + y = 18$

3. $5x - 2y = -10$

4. $3x - 8y = -21$

5. $4x - 9y = 16$

6. $15x - 8y = 5$

7. $6x - 10y = -8$

8. $14x - 6y = 0$

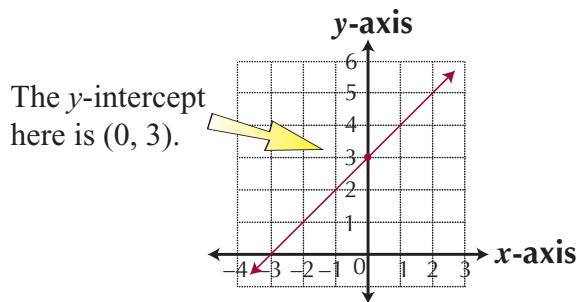
The y -Intercept is Where the Graph Crosses the y -Axis

The **y -axis** on a graph is the vertical line through the origin.

Every point on it has an x -coordinate of 0.

That means that all points on the y -axis are of the form **(0, y)**.

The **y -intercept** of the graph of $Ax + By = C$ is the **point** at which the graph of $Ax + By = C$ **crosses the y -axis**.



Computing the y -Intercept Using “ $x = 0$ ”

Since the y -intercept has an **x -coordinate of 0**, find the y -coordinate by letting **$x = 0$** in the equation of the line.

Check it out:

Always write the y -intercept as a point, not just as the value of y where the graph crosses the y -axis. For example, $(0, 3)$, not 3.

Example 3

Find the y -intercept of the line $-2x - 3y = -9$.

Solution

Let $x = 0$, then solve for y :

$$-2x - 3y = -9$$

$$-2(0) - 3y = -9$$

$$0 - 3y = -9$$

$$-3y = -9$$

$$y = 3$$

So **$(0, 3)$ is the y -intercept of $-2x - 3y = -9$.**

Example 4

Find the y -intercept of the line $3x + 4y = 24$.

Solution

Let $x = 0$, then solve for y :

$$3x + 4y = 24$$

$$3(0) + 4y = 24$$

$$0 + 4y = 24$$

$$4y = 24$$

$$y = 6$$

So **$(0, 6)$ is the y -intercept of $3x + 4y = 24$.**

Guided Practice

In Exercises 9–16, find the y -intercept.

9. $4x - 6y = 24$

10. $5x + 8y = 24$

11. $8x + 11y = -22$

12. $9x + 4y = 48$

13. $6x - 7y = -28$

14. $10x - 12y = 6$

15. $3x + 15y = -3$

16. $14x - 5y = 0$

Independent Practice

1. Define the x -intercept.
2. Define the y -intercept.

Find the x - and y -intercepts of the following lines:

- | | |
|------------------------|--|
| 3. $x + y = 9$ | 4. $x - y = 7$ |
| 5. $-x - 2y = 4$ | 6. $x - 3y = 9$ |
| 7. $3x - 4y = 24$ | 8. $-2x + 3y = 12$ |
| 9. $-5x - 4y = 20$ | 10. $-0.2x + 0.3y = 1$ |
| 11. $0.25x - 0.2y = 2$ | 12. $-\frac{1}{2}x - \frac{2}{3}y = 6$ |

13. $(\frac{3}{5}g, 0)$ is the x -intercept of the line $-10x - 3y = 12$.

Find the value of g .

14. $(0, \frac{1}{5}k)$ is the y -intercept of the line $2x - 15y = -3$.

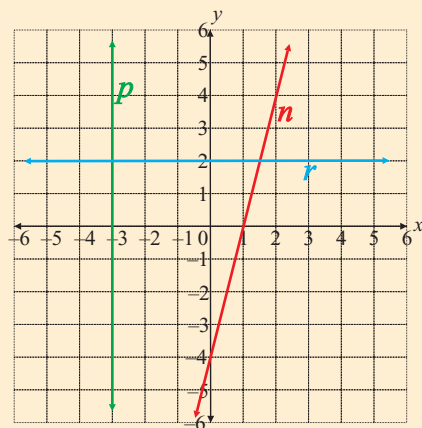
Find the value of k .

15. The point $(-3, b)$ lies on the line $2y - x = 8$. Find the value of b .

16. Find the x -intercept of the line in Exercise 15.

17. Another line has x -intercept $(4, 0)$ and equation $2y + kx = 20$. Find the value of k .

In Exercises 18-22, use the graph below to help you reach your answer.



18. Find the x - and y -intercepts of line n .

19. Find the x -intercept of line p .

20. Find the y -intercept of line r .

21. Explain why line p does not have a y -intercept.

22. Explain why line r does not have an x -intercept.

Round Up

Make sure you get the method the right way around — to find the x -intercept, put $y = 0$ and solve for x , and to find the y -intercept, put $x = 0$ and solve for y . In the next Topic you'll see that the intercepts are really useful when you're graphing lines from the line equation.

Graphing Lines

California Standards:

6.0: Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

7.0: Students verify that a point lies on a line, given an equation of the line.

Students are able to derive linear equations by using the point-slope formula.

What it means for you:

You'll graph lines by first calculating the x - and y -intercepts.

Key words:

- linear equation
- intercept

In Topic 4.2.2 you learned how to graph a straight line by plotting two points. If you're not given points on the line, it's easiest to use the x - and y -intercepts.

Graphing Lines by Computing the Intercepts

The method below for plotting a **straight-line graph** is the same as in Topic 4.2.2. To graph the line, you **plot two points** — except this time you use the **x -intercept** and the **y -intercept**, then draw a **straight line** through them.

Graphing a Line

- **Find the x -intercept** — let $y = 0$, then solve the equation for x .
- **Find the y -intercept** — let $x = 0$, then solve the equation for y .
- Draw a set of axes and **plot the two intercepts**.
- Draw a **straight line** through the points.
- **Check** your line by plotting a **third** point.

Example 1

Draw the graph of $5x + 3y = 15$ by computing the intercepts.

Solution

x -intercept:

$$5x + 3(0) = 15$$

$$5x + 0 = 15$$

$$5x = 15$$

$$x = 3$$

y -intercept:

$$5(0) + 3y = 15$$

$$0 + 3y = 15$$

$$3y = 15$$

$$y = 5$$

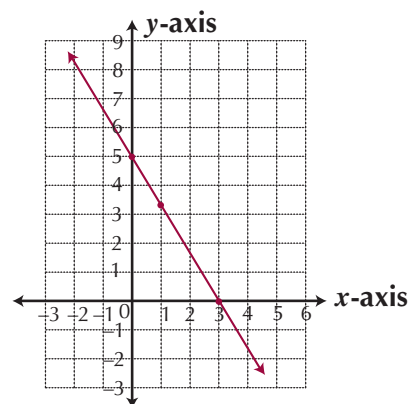
Therefore **$(3, 0)$ is the x -intercept** and **$(0, 5)$ is the y -intercept**.

Check: $x = 1$

$$5x + 3y = 15$$

$$\Rightarrow y = 5 - \frac{5}{3}x = 5 - \frac{5 \times 1}{3} = \frac{15 - 5}{3} = \frac{10}{3} = 3\frac{1}{3}$$

$\left(1, 3\frac{1}{3}\right)$ lies on the line — which means **the line is correct**.



Finding the intercepts is the quickest way of finding two points. When you substitute 0 for y to solve for x , the y -term disappears, and vice versa — making the equations easier to solve.

Example 2

Draw the graph of $y = -x + 3$ by computing the intercepts.

Solution

x-intercept:

$$0 = -x + 3$$

$$x = 3$$

y-intercept:

$$y = -(0) + 3$$

$$y = 3$$

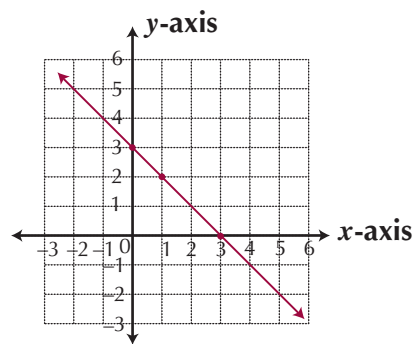
Therefore **(3, 0) is the x-intercept** and **(0, 3) is the y-intercept**.

Check: $x = 1$

$$y = -x + 3$$

$$\Rightarrow y = -1 + 3 = 2$$

(1, 2) lies on the line — which means **the line is correct**.



✓ Guided Practice

Draw the graphs of the following equations by computing the intercepts.

1. $5x + 2y = 10$

2. $-3x - 3y = 12$

3. $6x - y = 3$

4. $-4x + 5y = 20$

5. $2x + y = 3$

6. $-x - 8y = 2$

7. $x + y = 10$

8. $x - y = 4$

9. $2x - y = 4$

10. $x + 5y = 10$

11. $3x + y = 9$

12. $x - 4y = 8$

13. $y = 2x + 4$

14. $y = 5x - 10$

15. $y = 3x + 9$

16. $y = \frac{1}{4}x - 8$

17. $y = -\frac{2}{5}x - 2$

18. $y = \frac{4}{5}x + 4$

Independent Practice

Draw graphs of the lines using the x - and y -intercepts in Exercises 1–6.

- | | |
|------------------------------|---------------------------|
| 1. x -intercept: $(-3, 0)$ | y -intercept: $(0, 2)$ |
| 2. x -intercept: $(1, 0)$ | y -intercept: $(0, 6)$ |
| 3. x -intercept: $(4, 0)$ | y -intercept: $(0, -3)$ |
| 4. x -intercept: $(-6, 0)$ | y -intercept: $(0, -4)$ |
| 5. x -intercept: $(-1, 0)$ | y -intercept: $(0, 7)$ |
| 6. x -intercept: $(2, 0)$ | y -intercept: $(0, -5)$ |

Draw the graphs of the equations in Exercises 7–18 by computing the intercepts.

- | | |
|---------------------|----------------------|
| 7. $x + y = 6$ | 8. $x + y = -4$ |
| 9. $x - y = -5$ | 10. $x - y = 7$ |
| 11. $3x + y = 6$ | 12. $2x + y = 8$ |
| 13. $2x - y = -4$ | 14. $3x - y = -3$ |
| 15. $4x + 3y = -12$ | 16. $5x - 2y = 10$ |
| 17. $6x - 3y = 24$ | 18. $10x - 12y = 60$ |

19. Show that the graphs of $x + y = 6$ and $-6x - 6y = -36$ are the same.
20. Explain why the graph of $5x + 8y = 0$ cannot be drawn using the intercepts.

Round Up

*This Topic follows on neatly from Topic 4.2.2, where you graphed lines by plotting two points and joining them with a **straight line**. You can use any two points — the main reason for using the intercepts is that they're usually easier to calculate.*

Topic 4.3.1

Section 4.3 Slope of a Line

California Standards:

7.0: Students verify that a point lies on a line, given an equation of the line.

Students are able to derive linear equations by using the point-slope formula.

What it means for you:

You'll find the slope of a line given any two points on the line.

Key words:

- slope
- steepness
- horizontal
- vertical
- rise over run

Check it out:

Δx is pronounced “delta x ” and just means “change in x .”

By now you've had plenty of practice in plotting lines.

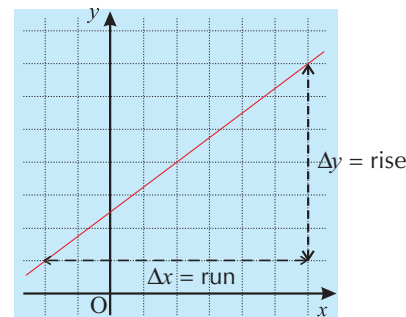
*Any line can be described by its **slope** — which is what this Topic is about.*

The Slope of a Line is Its Steepness

The **slope** (or **gradient**) of a line is a measure of its **steepness**.

The slope of a straight line is the **ratio** of the **vertical change** to the **horizontal change** between any two points lying on the line.

The vertical change is usually written Δy , and it's often called the **rise**. In the same way, the horizontal change is usually written Δx , and it's often called the **run**.



$$\text{Slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x}, \text{ provided } \Delta x \neq 0$$

If you know the coordinates of any **two points on a line**, you can find the slope. The slope, m , of a line passing through points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by this formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}, \text{ provided } x_2 - x_1 \neq 0$$

There is an important difference between **positive** and **negative** slopes — a **positive** slope means the line goes “**uphill**” (↗), whereas a line with a **negative** slope goes “**downhill**” (↘).

Use the Formula to Find the Slope of a Line

Example 1

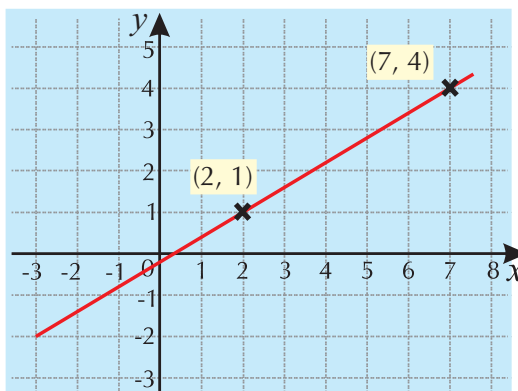
Find the slope of the line that passes through the points $(2, 1)$ and $(7, 4)$ and draw the graph.

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{7 - 2} = \frac{3}{5}$$

So the slope is $\frac{3}{5}$.

You know that the line passes through $(2, 1)$ and $(7, 4)$, so just join those two points up to draw the graph.



In the graph above, notice how the line has a **positive** slope, meaning it goes “uphill” from left to right.

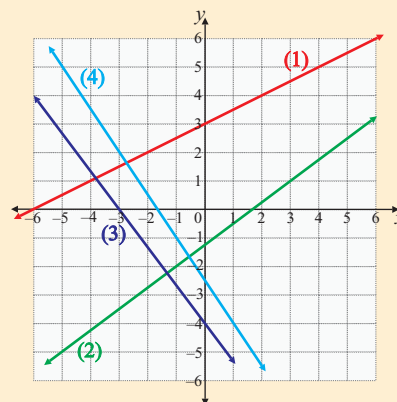
In fact, since the slope is $\frac{3}{5}$, the line goes **3 units up** for every **5 units across**.

Guided Practice

In Exercises 1–4, find the slope of the line on the graph opposite.

5. Find the slope of the line that passes through the points $(1, 5)$ and $(3, 2)$, and draw the graph.

6. Find the slope of the line that passes through the points $(3, 1)$ and $(2, 4)$, and draw the graph.



Be Careful If There are Negative Coordinates

Example 2

Find the slope of the line that passes through the points (3, 4) and (6, -2).

Solution

$$m = \frac{4 - (-2)}{3 - 6} = \frac{4 + 2}{-3} = \frac{6}{-3} = -2$$

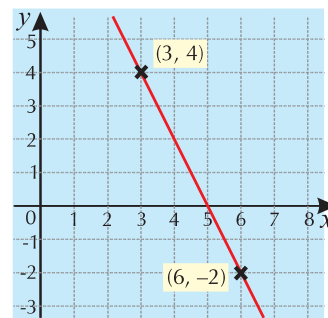
So the slope is **-2**.

Check it out:

It doesn't matter which point you call (x_1, y_1) and which you call (x_2, y_2) — choose whichever makes the math easier.

This time the line has a **negative** slope, meaning it goes “downhill” from left to right.

Here the slope is -2 , which means that the line goes **2 units down** for every **1 unit across**.



You Don't Have to Draw a Graph

Example 3

Find the slope of the lines through: a) (2, 5) and (-4, 2)
b) (1, -6) and (3, 3)

Solution

$$\text{a) } m = \frac{5 - 2}{2 - (-4)} = \frac{3}{2 + 4} = \frac{3}{6} = \frac{1}{2}$$

$$\text{b) } m = \frac{3 - (-6)}{3 - 1} = \frac{3 + 6}{2} = \frac{9}{2}$$

✓ Guided Practice

Find the slope m of the line through each pair of points below.

7. (-1, 2) and (3, 2)

8. (0, -5) and (-6, 1)

9. (5, -7) and (-3, -7)

10. (4, -1) and (-3, 5)

11. (-1, -3) and (1, -4)

12. (5, 7) and (-11, -12)

13. (-2, -2) and (-3, -17)

14. (18, 2) and (-32, 7)

15. (0, -1) and (1, 0)

16. (0, 0) and (-14, -1)

Some Problems Involve Variables

Example 4

If the slope of the line that passes through the points $(4, -1)$ and $(6, 2k)$ is 3, find the value of k .

Solution

Even though one pair of coordinates contains a variable, k , you still use the slope formula in exactly the same way as before.

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \text{ which means that } m = \frac{2k - (-1)}{6 - 4} = \frac{2k + 1}{2}$$

$$\begin{aligned} \text{But the slope is 3, so } \frac{2k + 1}{2} &= 3 \\ \Rightarrow 2k + 1 &= 6 \\ \Rightarrow 2k &= 5 \\ \Rightarrow k &= \frac{5}{2} \end{aligned}$$

✓ Guided Practice

Find the slope m of the lines through the points below.

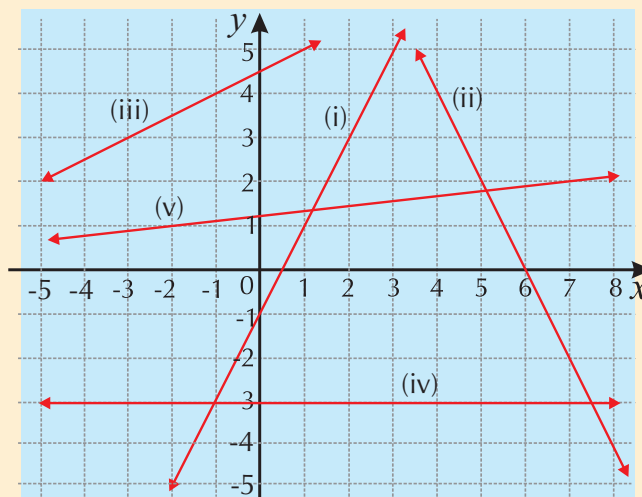
- | | |
|----------------------------------|----------------------------------|
| 17. $(7, -2c)$ and $(10, -c)$ | 18. $(b, 1)$ and $(3b, -3)$ |
| 19. $(2, 2k)$ and $(-5, -5k)$ | 20. $(3q, 1)$ and $(2q, 7)$ |
| 21. $(3d, 7d)$ and $(5d, 9d)$ | 22. $(4a, 5k)$ and $(2a, 7k)$ |
| 23. $(9c, 12v)$ and $(12c, 15v)$ | 24. (p, q) and (q, p) |
| 25. $(10, 14d)$ and $(d, -7)$ | 26. $(2t, -3s)$ and $(18s, 14t)$ |

In Exercises 27–32 you're given two points on a line and the line's slope, m . Find the value of the unknown constant in each Exercise.

27. $(-2, 3)$ and $(3k, -4)$, $m = \frac{2}{5}$
28. $(4, -5t)$ and $(7, -8t)$, $m = \frac{2}{7}$
29. $(4b, -6)$ and $(7b, -10)$, $m = \frac{3}{4}$
30. $(-8, -6)$ and $(12, 4)$, $m = -\frac{2}{5}v$
31. $(7k, -3)$ and $(k, -1)$, $m = 4$
32. $(1, -17)$ and $(40, 41)$, $m = -\frac{174}{78}t$

✓ Independent Practice

1. Find the slope m of the lines shown below.



In Exercises 2–5, find the slope of the line that passes through the given points, and draw the graph.

2. $(-2, 1)$ and $(0, 2)$

3. $(4, 4)$ and $(1, 0)$

4. $(-5, 2)$ and $(-1, 3)$

5. $(3, -3)$ and $(7, 3)$

In Exercises 6–10, find the slope of the line through each of the points.

6. $(-3, 5)$ and $(2, 1)$

7. $(0, 4)$ and $(-4, 0)$

8. $(2, 3)$ and $(4, 3)$

9. $(6d, 2)$ and $(4d, -1)$

10. $(2s, 2t)$ and $(s, 3t)$

In Exercises 11–15, you're given two points on a line and the line's slope, m . Find the value of the unknown constant in each Exercise.

11. $(3t, 7)$ and $(5t, 9)$, $m = -\frac{1}{2}$

12. $(3k, 1)$ and $(2k, 7)$, $m = \frac{1}{3}$

13. $(0, 14d)$ and $(10, -6d)$, $m = -1$

14. $(2t, -3)$ and $(-3t, 5)$, $m = \frac{5}{4}$

15. $(0, 8d)$ and $(-1, 4d)$, $m = -\frac{1}{3}$

Round Up

Slope is a measure of how *steep* a line is — it's *how many units up or down you go for each unit across*. If you go up or down a lot of units for each unit across, the line will be *steep* and the slope will be *large* (either large and positive if it goes up from left to right, or large and negative if it goes down from left to right).

Point-Slope Formula

California Standards:

7.0: Students verify that a point lies on a line, given an equation of the line.

Students are able to derive linear equations by using the point-slope formula.

What it means for you:

You'll learn about the point-slope formula and use it to find the equation of a line.

Key words:

- slope
- point-slope formula

Check it out:

The point-slope formula is just the one from Topic 4.3.1 rearranged. (x, y) is any point on the line, so (x_1, y_1) and (x, y) are the two points on the line.

The *point-slope formula* is a really useful way of calculating the equation of a straight line.

Use the Formula to Find the Equation of a Straight Line

If you know the **slope** of the line and a **point** on the line, you can use the **point-slope formula** to find the equation of the line.

The point-slope formula for finding the equation of a line is:

$$y - y_1 = m(x - x_1) \quad \text{where } m \text{ is the slope and } (x_1, y_1) \text{ is a point on the line.}$$

You substitute the **x-coordinate** of a point on the line for x_1 and the **y-coordinate** of the same point for y_1 . Watch out though — x and y are variables and they stay as letters in the equation of the line.

Example 1

Find the equation of the line through $(-4, 6)$ that has a slope of -3 .

Solution

$$(x_1, y_1) = (-4, 6) \text{ and } m = -3$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ \Rightarrow y - 6 &= -3[x - (-4)] \\ \Rightarrow y - 6 &= -3(x + 4) \\ \Rightarrow y - 6 &= -3x - 12 \\ \Rightarrow y + 3x &= -6 \end{aligned}$$

✓ Guided Practice

Write the equation of the line that passes through the given point and has the given slope.

1. Point $(-2, -3)$, slope $= -1$

2. Point $(3, -5)$, slope $= 2$

3. Point $(-7, -2)$, slope $= -5$

4. Point $(4, -3)$, slope $= \frac{2}{3}$

5. Point $(2, 6)$, slope $= -\frac{3}{4}$

6. Point $(-2, -3)$, slope $= \frac{5}{8}$

7. Point $(-5, -3)$, slope $= -\frac{6}{7}$

8. Point $\left(-\frac{2}{3}, \frac{1}{4}\right)$, slope $= \frac{2}{5}$

If Two Points are Given, Find the Slope First

If you know the coordinates of **two points** on a straight line, you can still find the equation using the **point-slope formula** — but you have to find the slope first.

Example 2

Write the equation of the straight line that contains the points $(3, -2)$ and $(-1, 5)$.

Solution

Step 1: Find the slope using the given points.

$$\text{Slope, } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-2)}{-1 - 3} = \frac{5 + 2}{-1 - 3} = -\frac{7}{4}$$

Step 2: Write the equation $y - y_1 = m(x - x_1)$

$$y - (-2) = -\frac{7}{4}(x - 3)$$

$$\Rightarrow y + 2 = -\frac{7}{4}(x - 3)$$

$$\Rightarrow 4y + 8 = -7(x - 3)$$

$$\Rightarrow 4y + 8 = -7x + 21$$

$$\Rightarrow \mathbf{4y + 7x = 13}$$

Check it out

You can use either point as (x_1, y_1) here — you'll still get the same equation.

✓ Guided Practice

Write the equation of the line that passes through the given pair of points.

9. $(-1, 0)$ and $(3, -4)$

11. $(-5, 7)$ and $(3, 9)$

13. $(8, 7)$ and $(-7, -5)$

15. $(3, 1)$ and $(5, 4)$

17. $(6, 2)$ and $(4, 1)$

19. $(1, 0)$ and $(2, 0)$

21. $(4, -1)$ and $(-1, -3)$

23. $(4, 5)$ and $(2, 6)$

25. $(3, -5)$ and $(0, 8)$

10. $(-1, 1)$ and $(-3, -3)$

12. $(6, -8)$ and $(-2, -10)$

14. $(-10, 11)$ and $(5, -12)$

16. $(2, -5)$ and $(3, -1)$

18. $(-3, 5)$ and $(4, 3)$

20. $(-2, -2)$ and $(7, 5)$

22. $(7, 2)$ and $(3, 3)$

24. $(-2, -3)$ and $(-3, -2)$

26. $(1, 1)$ and $(4, -6)$

✓ Independent Practice

In Exercises 1–6, write the equation of the line that passes through the given point and has the given slope.

- Point (1, 5), slope = -3
- Point (2, 0), slope = $\frac{1}{2}$
- Point (3, 1), slope = $\frac{1}{4}$
- Point (–3, 4), slope = $-\frac{2}{3}$
- Point (–8, 6), slope = $-\frac{4}{5}$
- Point (–3, –4), slope = $\frac{3}{8}$

In Exercises 7–16, write the equation of the line that passes through the given pair of points.

- (0, 3) and (4, –1)
- (–1, 6) and (7, 5)
- (3, 8) and (4, 4)
- (4, –7) and (–3, 5)
- (–6, 9) and (–4, –6)
- (4, –9) and (–3, –9)
- (–4, –8) and (–5, 4)
- (–8, 3) and (8, 4)
- (10, 5) and (4, 6)
- (0, 0) and (–4, –6)

17. The points (5, 6) and (8, 7) lie on a line.

Find the equation of this line.

18. The line in Exercise 17 forms one side of a triangle that has one vertex at the point (5, –4). If the slope of one of the edges of the triangle is -3 , find the equation of this edge.

19. The point (8, k) lies on the third edge of the triangle in Exercise 18. Given that the triangle is isosceles, find, by graphing, the value of k .

20. Joshua is an architect who must build a wheelchair-accessible office building. To make a ramp that is easy to maneuver in a wheelchair, Joshua designs a ramp that is 27 inches high and 540 inches long. What is the slope of the ramp?

21. If the student population at a high school changes from 1372 in 1996 to 1768 in 2006, what is the average rate of change of the student population?

(Hint: Use the pairs of coordinates (1996, 1372) and (2006, 1768) to reach your answer.)

Round Up

You should look over the *point-slope formula* until you can write it down from memory. It's a really useful formula and makes *finding the equation of a line* much easier — but only if you remember it.

Topic 4.4.1

Section 4.4 Parallel Lines

California Standards:

8.0: Students understand the concepts of parallel lines and perpendicular lines and how their slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

What it means for you:

You'll work out the slopes of parallel lines and you'll test if two lines are parallel.

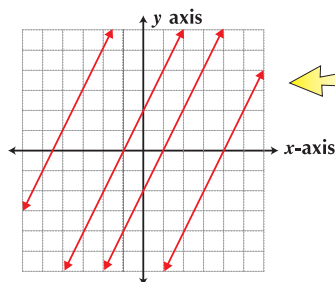
Key words:

- parallel
- intersect

Now that you've practiced finding the *slope* of a line, you can use the method on a special case — *parallel lines*.

Parallel Lines Never Meet

Parallel lines are two or more lines in a plane that **never intersect** (cross).



These lines are all parallel. No matter how long you draw them, they'll never meet.

The symbol \parallel is used to indicate parallel lines — you read this symbol as “**is parallel to**.” So, if l_1 and l_2 are lines, then $l_1 \parallel l_2$ means “line l_1 is parallel to line l_2 .”

Parallel Lines Have Identical Slopes

You can determine whether lines are **parallel** by looking at their **slopes**.

Two lines are parallel if their **slopes are equal**.

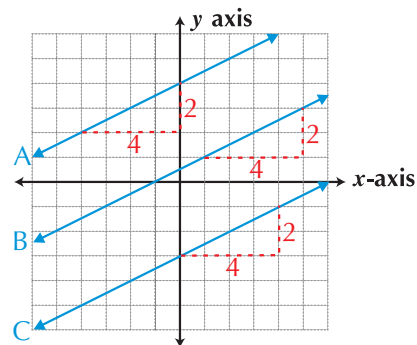
Example 1

Prove that the three lines A, B, and C shown on the graph are parallel.

Solution

Using the rise over run formula (see Topic 4.3.1), you can see that

they all have a slope of $\frac{2}{4} = \frac{1}{2}$.



✓ Guided Practice

- Two lines on the same plane that never intersect are called _____ lines.
- To determine if two lines are parallel you can look at their _____.
- Prove that the line f defined by $y - 3 = \frac{2}{3}(x - 4)$ is parallel to line g defined by $y - 6 = \frac{2}{3}(x + 1)$.

Vertical Lines Don't Have Defined Slopes

Vertical lines are **parallel**, but you can't include them in the definition on page 197 because their slopes are undefined.

Points on a vertical line all have the **same x-coordinate**, so they are of the form (c, y_1) and (c, y_2) . The slope of a vertical line is undefined because

$$m = \frac{y_2 - y_1}{c - c} = \frac{y_2 - y_1}{0} \text{ is not defined.}$$

Check it out:

The slope of a vertical line is undefined because division by 0 is undefined.

Don't forget:

See Topic 4.3.1 for more on calculating the slope of a line.

Check it out:

It's possible that all the points lie on the same line — meaning there is actually only one line. To be absolutely accurate, you should check that this isn't the case before you say the lines are parallel. You can do this by finding the equations of the lines and comparing them.

Test if Lines are Parallel by Finding Slopes

To check if a pair of lines are **parallel**, just find the **slope** of each line.

If the slopes are **equal**, the lines are **parallel**.

Example 2

Show that the straight line through $(2, -3)$ and $(-5, 1)$ is parallel to the straight line joining $(7, -1)$ and $(0, 3)$.

Solution

Step 1: Find the slope of each line using the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-3)}{-5 - 2} = \frac{4}{-7} = -\frac{4}{7}$$

$$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-1)}{0 - 7} = \frac{4}{-7} = -\frac{4}{7}$$

Step 2: Compare the slopes and draw a conclusion.

$$-\frac{4}{7} = -\frac{4}{7}, \text{ so } m_1 = m_2.$$

So the straight line through $(2, -3)$ and $(-5, 1)$ is parallel to the straight line through $(7, -1)$ and $(0, 3)$.

✓ Guided Practice

4. Show that line a , which goes through points $(7, 2)$ and $(3, 3)$, is parallel to line b joining points $(-8, -4)$ and $(-4, -5)$.
5. Show that the line through points $(4, 3)$ and $(-1, 3)$ is parallel to the line through points $(-6, -1)$ and $(-8, -1)$.
6. Determine if line f joining points $(1, 4)$ and $(6, 2)$ is parallel to line g joining points $(0, 8)$ and $(10, 4)$.
7. Determine if the line through points $(-5, 2)$ and $(3, 7)$ is parallel to the line through points $(-5, 1)$ and $(-3, 6)$.
8. Determine if the line through points $(-8, 4)$ and $(-8, 3)$ is parallel to the line through points $(6, 3)$ and $(-4, 3)$.

Some Parallel Line Problems are Tougher

Example 3

Find the equation of a line through $(-1, 4)$ that is parallel to the straight line joining $(5, 7)$ and $(-6, -8)$.

Solution

Step 1: Find the slope m_1 of the line through $(5, 7)$ and $(-6, -8)$.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-8 - 7}{-6 - 5} = \frac{-15}{-11} = \frac{15}{11}$$

Step 2: The slope m_2 of the line through $(-1, 4)$ must be equal to $\frac{15}{11}$ since the lines are parallel.

$$\text{So, } m_1 = m_2 = \frac{15}{11}$$

Step 3: Now use the point-slope formula to find the equation of the line through point $(-1, 4)$ with slope $\frac{15}{11}$.

$$\begin{aligned}y - y_1 &= m(x - x_1) \\ \Rightarrow y - 4 &= \frac{15}{11}[x - (-1)] \\ \Rightarrow 11y - 44 &= 15(x + 1) \\ \Rightarrow 11y - 44 &= 15x + 15\end{aligned}$$

$$\text{Equation: } \mathbf{11y - 15x = 59}$$

Check it out:

This is the equation of the line through $(-1, 4)$ that is parallel to the straight line joining $(5, 7)$ and $(-6, -8)$.

✓ Guided Practice

- Find the equation of the line through $(-3, 7)$ that is parallel to the line joining points $(4, 5)$ and $(-2, -8)$.
- Find the equation of the line through $(6, -4)$ that is parallel to the line joining points $(-1, 6)$ and $(7, 3)$.
- Find the equation of the line through $(-1, 7)$ that is parallel to the line joining points $(4, -3)$ and $(8, 6)$.
- Write the equation of the line through $(-3, 5)$ that is parallel to the line joining points $(-1, 2.5)$ and $(0.5, 1)$.
- Write the equation of the line through $(-2, -1)$ that is parallel to the line $x + 3y = 6$.

✓ Independent Practice

- Line l_1 has slope $\frac{1}{2}$ and line l_2 has slope $\frac{1}{2}$. What can you conclude about l_1 and l_2 ?
- Line l_1 has a slope of $-\frac{1}{3}$. If $l_1 \parallel l_2$, then what is the slope of l_2 ?
- Show that all horizontal lines are parallel.
- Show that the line through the points $(5, -3)$ and $(-8, 1)$ is parallel to the line through $(13, -7)$ and $(-13, 1)$.
- Determine if the line through the points $(5, 4)$ and $(0, 9)$ is parallel to the line through $(-1, 8)$ and $(4, 0)$.
- Determine if the line through the points $(-2, 5)$ and $(6, 0)$ is parallel to the line through $(8, -1)$ and $(0, 4)$.
- Determine if the line through the points $(4, -7)$ and $(4, -4)$ is parallel to the line through $(-5, 1)$ and $(-5, 5)$.
- Determine if the line through the points $(-2, 3)$ and $(-2, -2)$ is parallel to the line through $(1, 7)$ and $(-6, 7)$.
- Find the equation of the line through $(1, -2)$ that is parallel to the line joining the points $(-3, -1)$ and $(8, 7)$.
- Find the equation of the line through $(-5, 3)$ that is parallel to the line joining the points $(-2, 6)$ and $(8, -1)$.
- Write the equation of the line through $(0, 6)$ that is parallel to the line $3x + 2y = 6$.

Round Up

When you draw lines with *different slopes* on a set of axes, you might not see where they cross. But remember, you are only looking at a tiny bit of the lines — they go on *indefinitely* in both directions. If they don't have identical slopes, they'll *cross* sooner or later.

Topic 4.4.2

Perpendicular Lines

California Standards:

8.0: Students understand the concepts of parallel lines and **perpendicular lines** and how their slopes are related. Students are able to find the equation of a line **perpendicular to a given line that passes through a given point.**

What it means for you:

You'll work out the slopes of perpendicular lines and you'll test if two lines are perpendicular.

Key words:

- perpendicular
- reciprocal

Check it out:

This definition doesn't work for horizontal or vertical lines, since the slope of a vertical line can't be defined in the same way as for other lines. But remember that the lines $x = c$ and $y = k$ (for constants c and k) are perpendicular.

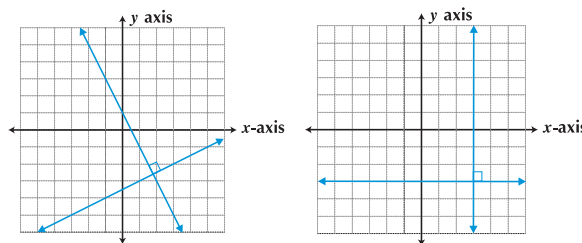
Don't forget:

See Topic 4.3.1 for the "rise over run" formula.

Math problems about parallel lines often deal with **perpendicular lines** too. "Perpendicular" might sound like a difficult term, but it's actually a really simple idea.

Perpendicular Lines Meet at Right Angles

Two lines are **perpendicular** if they intersect at **90° angles**, like in the graphs on the right.



Slopes of Perpendicular Lines are Negative Reciprocals

Two lines are **perpendicular** if the slope of one is the **negative reciprocal** of the slope of the other.

To get the reciprocal of a number you divide 1 by it.

For example, the reciprocal of x is $\frac{1}{x}$ and so the negative reciprocal is $-\frac{1}{x}$.

The reciprocal of $\frac{x}{y}$ is $\frac{1}{\frac{x}{y}} = \frac{y}{x}$ and so the negative reciprocal is $-\frac{y}{x}$.

Example 1

Prove that lines A and B, shown on the graph, are perpendicular to each other.

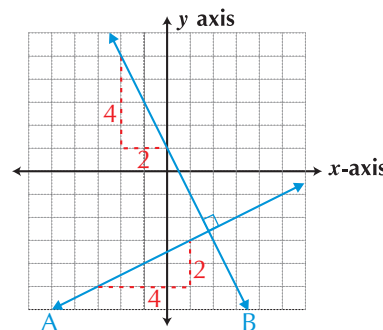
Solution

Using the rise over run formula:

$$\text{Slope of A} = m_1 = \frac{2}{4} = \frac{1}{2}$$

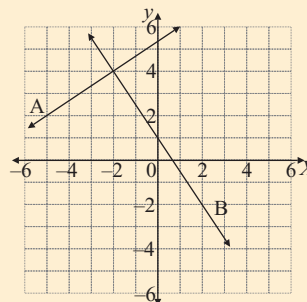
$$\text{Slope of B} = m_2 = \frac{-4}{2} = -2$$

$\frac{1}{2}$ is the **negative reciprocal** of -2 , so A and B must be perpendicular.



✓ Guided Practice

1. Perpendicular lines meet at _____ angles.
2. Find the negative reciprocal of 3.
3. Find the negative reciprocal of $-\frac{1}{4}$.
4. Find the negative reciprocal of $-\frac{4}{5}$.
5. Use the graph to prove that A and B are perpendicular.



Perpendicular Lines: $m_1 \times m_2 = -1$

When you **multiply a number by its reciprocal**, you always get 1.

For example, $\frac{1}{5} \times 5 = \frac{5}{5} = 1$ and $\frac{3}{2} \times \frac{2}{3} = 1$.

So because two perpendicular slopes are **negative reciprocals** of each other, their **product** is always -1 . Here's the same thing written in math-speak:

Check it out:

\perp is the symbol for "is perpendicular to."

If two lines l_1 and l_2 have slopes m_1 and m_2 ,
 $l_1 \perp l_2$ if and only if $m_1 \times m_2 = -1$.

Example 2

P and Q are two straight lines and $P \perp Q$. P has a slope of -4 . What is the slope of Q?

Solution

$$\begin{aligned}m_P \times m_Q &= -1 \\ \Rightarrow -4 \times m_Q &= -1 \\ \Rightarrow m_Q &= \frac{-1}{-4} = \frac{1}{4}.\end{aligned}$$

So the slope of Q is $\frac{1}{4}$.

✓ Guided Practice

6. Lines l_1 and l_2 are perpendicular. If the slope of l_1 is $\frac{1}{5}$, find the slope of l_2 .
7. Lines A and B are perpendicular. If the slope of A is $-\frac{5}{8}$, find the slope of B.
8. Lines R and T are perpendicular. If R has slope $-\frac{7}{11}$, what is the slope of T?
9. The slope of l_1 is -0.8 . The slope of l_2 is 1.25 . Determine whether l_1 and l_2 are perpendicular.

Show That Lines are Perpendicular by Finding Slopes

Example 3

Determine the equation of the line passing through $(3, 1)$ that is perpendicular to the straight line through $(2, -1)$ and $(4, 2)$.

Solution

Step 1: Find slope m_1 of the line through $(2, -1)$ and $(4, 2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-1)}{4 - 2} = \frac{3}{2}$$

Step 2: Find the slope m_2 of a line perpendicular to that line:

$$m_1 \times m_2 = -1$$

$$\frac{3}{2} \times m_2 = -1$$

$$\Rightarrow m_2 = -\frac{2}{3}$$

Step 3: Now use the point-slope formula to find the equation of the line through $(3, 1)$ with slope $-\frac{2}{3}$.

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow y - 1 = -\frac{2}{3}(x - 3)$$

$$\Rightarrow 3y - 3 = -2(x - 3)$$

$$\Rightarrow 3y - 3 = -2x + 6$$

Equation: $3y + 2x = 9$

✓ Guided Practice

10. Show that the line through the points $(5, -3)$ and $(-8, 1)$ is perpendicular to the line through $(4, 6)$ and $(8, 19)$.
11. Show that the line through $(0, 6)$ and $(5, 1)$ is perpendicular to the line through $(4, 8)$ and $(-1, 3)$.
12. Show that the line through $(4, 3)$ and $(2, 2)$ is perpendicular to the line through $(1, 3)$ and $(3, -1)$.
13. Determine the equation of the line through $(3, -4)$ that is perpendicular to the line through the points $(-7, -3)$ and $(-3, 8)$.
14. Determine the equation of the line through $(6, -7)$ that is perpendicular to the line through the points $(8, 2)$ and $(-1, 8)$.
15. Find the equation of the line through $(4, 5)$ that is perpendicular to the line $-3y + 4x = 6$.

✓ Independent Practice

In Exercises 1–8, J and K are perpendicular lines. The slope of J is given. Find the slope of K.

1. $m_J = -3$

2. $m_J = -14$

3. $m_J = \frac{5}{2}$

4. $m_J = \frac{6}{7}$

5. $m_J = -\frac{8}{3}$

6. $m_J = -\frac{14}{15}$

7. $m_J = -0.18$

8. $m_J = 0.45$

9. Show that the line through $(2, 7)$ and $(-2, 8)$ is perpendicular to the line through $(-3, -3)$ and $(-2, 1)$.
10. Show that the line through $(-4, 3)$ and $(3, -2)$ is perpendicular to the line through $(-7, -1)$ and $(-2, 6)$.
11. Determine the equation of the line through $(5, 9)$ that is perpendicular to a line with slope $\frac{1}{3}$.
12. Determine the equation of the line through $(3, -5)$ that is perpendicular to the line through the points $(-3, 2)$ and $(-6, -4)$.

Round Up

“Perpendicular” is just a special math word to describe lines that are at right angles to each other. Remember that the best way to show that two lines are at right angles is to calculate their slopes — if they multiply together to make -1 , then the lines are perpendicular.

The Slope-Intercept Form of a Line

California Standards:

8.0: Students understand the concepts of parallel lines and perpendicular lines and how their slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

What it means for you:

You'll plot graphs and solve equations using a method called the slope-intercept form of an equation.

Key words:

- slope-intercept form
- intercept

Don't forget:

In the slope-intercept form of the line equation, m and b are numbers, and x and y are variables.

Now that you've practiced calculating the slope and intercept of a line, you can use the two things together to make *plotting graphs* easier.

The Slope-Intercept Form of a Line: $y = mx + b$

In the slope-intercept form the **y is alone** on one side of the equation.

The **slope-intercept form** of the equation is:

$$y = mx + b$$

Here are a few examples of equations in the slope-intercept form:

$$y = 3x + 2 \quad y = -\frac{1}{2}x + 1 \quad y = -x + 1 \quad y = \frac{2}{3}x - 4$$

✓ Guided Practice

In Exercises 1–6, decide whether each equation is in slope-intercept form or not.

1. $y = 3x + 7$

2. $3x + 4y = 7$

3. $y - 3 = 2(x - 4)$

4. $y - 8 = 3(x - 4)$



5. $y = \frac{3}{2}x + 18$

6. $y = -4x - 1$

The Slope-Intercept Form Makes It Easy to Plot Graphs

The **slope-intercept form** of an equation is really useful for plotting graphs because m is the **slope** of the line and b is the **y -coordinate of the y -intercept**.

$$y = mx + b$$

slope   y -coordinate of y -intercept

Example 1

Plot the graph of $y = \frac{1}{2}x + 2$.

Solution

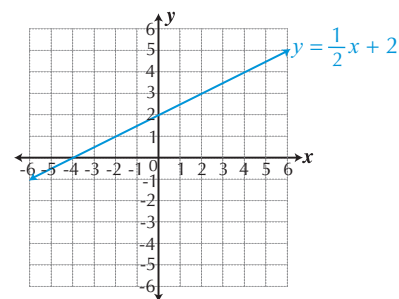
$$y = mx + b = \frac{1}{2}x + 2$$

$$\text{Slope} = m = \frac{1}{2}$$

— for a slope of $\frac{1}{2}$, go up 1 unit for every 2 units across.

y -coordinate of the y -intercept = $b = 2$

— so the y -intercept is $(0, 2)$.

**Example 2**

Plot the graph of $y = -3x - 4$.

Solution

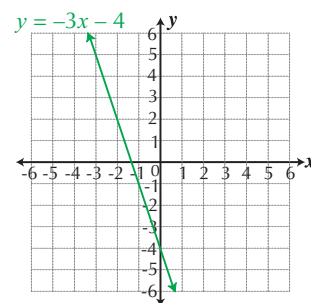
$$y = mx + b = -3x - 4$$

$$\text{Slope} = m = -3$$

— for a slope of -3 , go down 3 units for every unit across.

y -coordinate of the y -intercept = $b = -4$

— so the y -intercept is $(0, -4)$.

**Don't forget:**

You can write $y = -3x - 4$ as $y = -3x + (-4)$, so the y -intercept is -4 .

Guided Practice

7. In the equation $y = 8x + 5$, find the slope.

8. In the equation $y = -x + 10$, find the slope.

9. In the equation $y = 2x + 5$, find the y -intercept.

10. In the equation $y = 7b - 3$, find the y -intercept.

In Exercises 11–18, plot each equation on a graph.

11. $y = 2x + 3$

12. $y = x - 6$

13. $y = -7x - 8$

14. $y = -\frac{1}{3}x - 4$

15. $y = -\frac{1}{2}x + 6$

16. $y = \frac{1}{5}x - 3$

17. $y = \frac{3}{4}x$

18. $y = 6$

Solve for y to Get the Slope-Intercept Form

OK, so you know the **slope-intercept form** of an equation makes drawing graphs a lot easier. The trouble is, you'll often be given an equation which **isn't** in slope-intercept form.

To get the equation into slope-intercept form, **solve for y** .

Example 3

A line has the equation $Ax + By = C$, where $B \neq 0$.
Solve this equation for y , justifying each step.

Solution

$$\begin{aligned} Ax + By &= C && \text{Given equation} \\ Ax - Ax + By &= -Ax + C && \text{Subtraction property of equality} \\ By &= -Ax + C \\ \frac{By}{B} &= -\frac{Ax}{B} + \frac{C}{B} && \text{Division property of equality} \\ y &= -\frac{A}{B}x + \frac{C}{B} \end{aligned}$$

The equation is now in slope-intercept form. The slope, $m = -\frac{A}{B}$ and the y -coordinate of the y -intercept, $b = \frac{C}{B}$.

Example 4

Determine the slope and y -intercept of the line $2x - 3y = 9$.

Solution

Step 1: Solve the given equation for y .

$$\begin{aligned} 2x - 3y &= 9 \\ -3y &= -2x + 9 \\ y &= \frac{-2}{-3}x + \frac{9}{-3} \\ y &= \frac{2}{3}x - 3 \end{aligned}$$

Now you've got the equation in slope-intercept form, $y = mx + b$.

Step 2: Get the slope and y -intercept from the equation.

The **slope, $m = \frac{2}{3}$** .

The y -coordinate of the y -intercept, $b = -3$.
So, the **y -intercept = $(0, -3)$** .

✓ Guided Practice

In Exercises 19–24, find the slope and y -intercept of the line.

19. $3x + 3y = 9$

20. $2y - 6x = 10$

21. $2x - 2y = 5$

22. $-7y + 5x = 14$

23. $-2y + 3x = 8$

24. $-5x + 4y = 12$

Write down the equations of the following lines in slope-intercept form.

25. The line with slope 4 that passes through the point $(0, 2)$.

26. The line with slope 2 that passes through the point $(0, -6)$.

27. The line with slope -3 that passes through the point $(0, 1)$.

28. The line with slope $-\frac{6}{7}$ that passes through the point $(0, -3)$.

✓ Independent Practice

In Exercises 1–10, find the slope and y -intercept of each equation that's given.

1. $y = \frac{1}{2}x - 5$

2. $y = \frac{2}{3}x + 1$

3. $y = 3x + 6$

4. $-6y = 3x + 12$

5. $y = 2(x + 2)$

6. $y - 4 = 2(x + 1)$

7. $7 - y = 5(x + 4)$

8. $y - 3 = 2(x - 9)$

9. $3x + 4y = 8$

10. $2x + 3y = 9$

In Exercises 11–15, plot the graph of the given equation.

11. $y = \frac{1}{3}x + 5$

12. $y = -\frac{1}{3}x - 6$

13. $y = x + 2$

14. $y = -x + 2$

15. $y = 2x$

In Exercises 16–20, write the equations of the lines in slope-intercept form.

16. A line with slope $\frac{4}{3}$ that passes through the point $(0, 4)$

17. A line with slope $\frac{1}{2}$ that passes through the point $(0, -2)$

18. $4x + 2y = 8$

19. $6x - 3y = 15$

20. $3x - 4y = -16$

In Exercises 21–22, write the equations of the lines in slope-intercept form.

21. The line with slope 0 that passes through the point $(2, 6)$

22. The line with slope 2 that passes through the point $(6, 3)$

More About Slopes

California Standards:

8.0: Students understand the concepts of parallel lines and perpendicular lines and how their slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

What it means for you:

You'll learn how to tell whether lines are parallel or perpendicular by looking at the slope-intercept form.

Key words:

- parallel
- perpendicular
- reciprocal

Don't forget:

See Topic 4.3.1 for more on the slope of a line.

Don't forget:

Use the point-slope formula once you have all the information you need.

This Topic carries on from the material on [parallel](#) and [perpendicular lines](#) that you learned earlier in this Section.

Values of m Tell You if Lines are Parallel

Parallel lines all have the **same slope**, so the slope-intercept forms of their equations all have the **same value of m** .

For example, the lines $y = 3x + 2$, $y = 3x - 1$ and $y = 3x - 6$ are all parallel.

Example 1

Find the equation of the line through $(4, -4)$ that is parallel to the line $2x - 3y = 6$.

Solution

Step 1: Write $2x - 3y = 6$ in the slope-intercept form — that is, solve the equation for y .

$$2x - 3y = 6 \quad \Rightarrow \quad -3y = -2x + 6 \quad \Rightarrow \quad y = \frac{2}{3}x - 2$$

Step 2: Get the slope from the equation.

The slope of the line $y = \frac{2}{3}x - 2$ is $\frac{2}{3}$. Since the required line through $(4, -4)$ is parallel to the line $y = \frac{2}{3}x - 2$, its slope is also $\frac{2}{3}$.

Step 3: Now write the equation of the line through $(4, -4)$ with a slope of $\frac{2}{3}$.

$$\begin{aligned} y - y_1 &= m(x - x_1) &\Rightarrow & y - (-4) = \frac{2}{3}(x - 4) \\ & &\Rightarrow & 3(y + 4) = 2(x - 4) \\ & &\Rightarrow & 3y + 12 = 2x - 8 \\ & &\Rightarrow & \mathbf{3y - 2x = -20} \end{aligned}$$

Guided Practice

1. Give an example of a line that is parallel to $y = \frac{1}{2}x + 1$.
2. Is the line $y = \frac{4}{5}x - 2$ parallel to the line $y = \frac{4}{5}x + 6$? Explain.
3. Find the equation of the line through $(-4, 3)$ that is parallel to the line $y = 3x + 9$.
4. Find the equation of the line through $(3, 8)$ that is parallel to the line $3x + y = 1$.

Values of m Also Tell You if Lines are Perpendicular

Don't forget:

See Topic 4.4.2 for a reminder on perpendicular lines.

The slope-intercept forms of equations of perpendicular lines have values of m that are negative reciprocals of each other.

For example, the lines $y = 3x + 2$ and $y = -\frac{1}{3}x - 1$ must be perpendicular, because the negative reciprocal of 3 is $-\frac{1}{3}$.

Example 2

Find the equation of the line through $(2, -4)$ that is perpendicular to the line $-3y - x = 5$.

Solution

Step 1: Write $-3y - x = 5$ in the slope-intercept form (that is, solve the equation for y).

$$-3y - x = 5 \quad \Rightarrow \quad -3y = x + 5 \quad \Rightarrow \quad y = -\frac{1}{3}x - \frac{5}{3}$$

Step 2: Get the slope (m_1) of the line $y = -\frac{1}{3}x - \frac{5}{3}$ and determine the slope (m_2) of the required line through $(2, -4)$.

Since $m_1 = -\frac{1}{3}$, and m_2 is the negative reciprocal of $-\frac{1}{3}$, m_2 must be 3.

Step 3: Write the equation of the line through $(2, -4)$ with a slope of 3. Use the point-slope formula here:

$$\begin{aligned} y - y_1 &= m(x - x_1) &\Rightarrow & y - (-4) = 3(x - 2) \\ & &\Rightarrow & y + 4 = 3x - 6 \\ & &\Rightarrow & \mathbf{y - 3x = -10} \end{aligned}$$

Guided Practice

5. Give an example of a line that's perpendicular to the line $y = 6x$.
6. Is the line $y = 4x + 2$ perpendicular to the line $y = -\frac{1}{4}x + 4$? Explain your answer.
7. Find the equation of the line through $(-2, 0)$ that is perpendicular to the line $y = -2x - 4$.
8. Find the equation of the line through $(-4, 6)$ that is perpendicular to the line $3x - 4y = 24$.

✓ Independent Practice

Check it out:

“Collinear” means “on the same straight line.” If two lines have the same slope, and pass through the same point, they must be collinear.

In Exercises 1–8, determine whether the pairs of lines are parallel, perpendicular, or collinear.

1. $y = 2x + 1$ and $y = -\frac{1}{2}x - 6$

2. $y = \frac{1}{3}x + 5$ and $y = -3x - 4$

3. $y = 4x - 8$ and $y = -\frac{1}{4}x + 2$

4. $y = 6$ and $y = 3$

5. $x = 2$ and $y = -4$

6. $5x - 2y = -10$ and $10x - 4y = -20$

7. $3x + y = 6$ and $6x + 2y = -4$

8. $2x - y = -4$ and $6x - 3y = -12$

In Exercises 9–20, find the equations of the lines.

9. The line through $(5, 2)$ that's parallel to a line with slope $\frac{1}{2}$.

10. The line through $(3, -3)$ that's parallel to a line with slope $\frac{2}{5}$.

11. The line through $(2, -9)$ that's perpendicular to a line with slope -3 .

12. The line through $(-3, 1)$ that's perpendicular to a line with slope $\frac{5}{8}$.

13. The line through $(0, 0)$ that's parallel to $3x + y = 18$.

14. The line through $(3, 5)$ that's parallel to $3x - 7y = -21$.

15. The line through $(4, -3)$ that's parallel to $3x - 4y = 16$.

16. The line through $(-2, 6)$ that's parallel to $6x - 10y = -20$.

17. The line through $(0, 6)$ that's perpendicular to $2x + y = 18$.

18. The line through $(-3, -5)$ that's perpendicular to $3x - 6y = -24$.

19. The line through $(6, -2)$ that's perpendicular to $3x - 5y = -10$.

20. The line through $(8, 2)$ that's perpendicular to the line joining the points $(-6, 3)$ and $(-2, 6)$.

21. Use slopes to decide whether the points $(-3, -8)$, $(3, -2)$, and $(8, 3)$ are collinear (on the same line) or noncollinear.

22. Use slopes to decide whether the points $(4, 5)$, $(3, -2)$, and $(8, 3)$ are collinear or noncollinear.

23. Use slopes to decide whether the points $B(2, 10)$, $K(-1, 3)$, and $J(5, -3)$ are vertices of a right triangle.

24. Show that the points $M(3, 11)$, $A(-4, 4)$, and $T(3, -3)$ are vertices of a right triangle.

Round Up

Hopefully you'll see now why the slope-intercept form of a line is so useful — you can just glance at the equations to see whether lines are *parallel* or *perpendicular*, without having to plot the graphs.

Topic 4.5.1

Section 4.5 Regions Defined by Inequalities

California Standards:

6.0: Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

What it means for you:

You'll learn to sketch the region defined by a linear inequality.

Key words:

- inequality
- plane
- region

Just like with equations, you can graph inequalities on the coordinate plane. The only tricky bit is showing whether the solution set is above or below the line. This Topic will show you how.

A Line Divides the Plane into Three Regions

The graph of a **linear** equation divides the plane into **three** regions:

- The set of points that lie on the line.
- The set of points that lie above the line.
- The set of points that lie below the line.

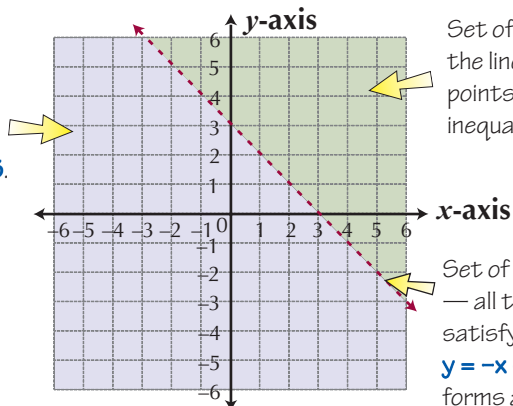
The regions **above** and **below** the line of a linear equation are each represented by a **linear inequality**.

Example 1

Graph $y = -x + 3$ and show that it divides the plane into three regions.

Solution

Set of points **below** the line — all these points satisfy the inequality $y < -x + 3$.



Set of points **above** the line — all these points satisfy the inequality $y > -x + 3$.

Set of points **on** the line — all these points satisfy the equation $y = -x + 3$. The line forms a **border** between the other two regions.

Check it out:

You'll find more about the dashed line in Topic 4.5.2.

The points on the line **don't** satisfy either of the inequalities — that's why the line is **dashed**.

Identifying the Region

To **identify** the inequality defining a region, choose a point from the region and **substitute** its coordinates into the equation of the line that borders the region. Since the point doesn't lie on the line, the equation won't be a true statement.

To **make the statement true**, you need to replace the “=” with a “<” or “>” sign. The **resulting inequality** defines the region containing the point.

Example 2

State the inequality that defines the shaded region.

Solution

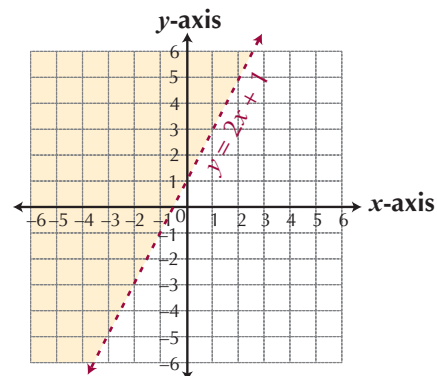
Choose a point in the shaded region, for example, (0, 2). Test this point in the equation of the line: $y = 2x + 1$

So at (0, 2), you get $2 = 2(0) + 1$, that is, $2 = 1$, which is a false statement.

Since $2 > 1$, a “>” sign is needed to make it a true statement.

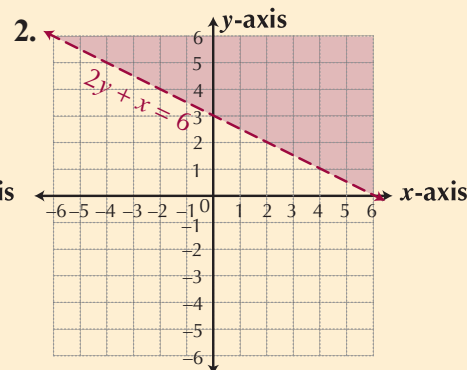
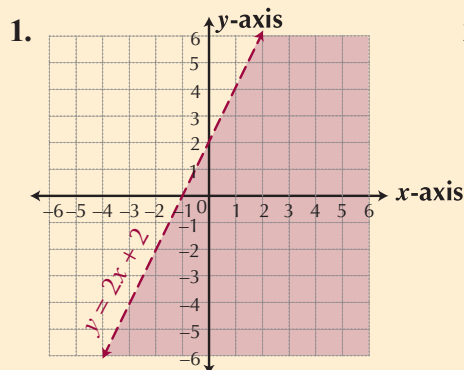
So (0, 2) satisfies the inequality $y > 2x + 1$.

Therefore the inequality that defines the shaded region is $y > 2x + 1$.



Guided Practice

In Exercises 1–2, state the inequality that defines the shaded region on each of the graphs.



Sketching the Region Defined by a Linear Inequality

An **ordered pair** (x, y) is a **solution** of a linear inequality if its x and y values **satisfy** the inequality.

Check it out:

Use whichever method you find easiest for sketching the line — see Section 4.2.

For example, you could plot the intercepts if you prefer.

The **graph** of a linear inequality is the **region** consisting of **all the solutions** of the inequality (the **solution set**).

To **sketch** the region defined by a linear inequality, you need to **plot the graph** of the corresponding equation (**the border line**), then **shade the correct region**. To decide which is the correct region, just **test a point**.

Example 3

Sketch the region defined by $6x - 3y < 9$.

Solution

First plot the graph of the corresponding equation. This is the border line.

Rearrange the equation into the form $y = mx + b$:

$$6x - 3y = 9 \Rightarrow 3y = 6x - 9 \Rightarrow y = 2x - 3$$

Table of values for sketching the line:

x	y	(x, y)
0	$y = 2x - 3 = 2(0) - 3 = -3$	$(0, -3)$
3	$y = 2x - 3 = 2(3) - 3 = 3$	$(3, 3)$

So the border line goes through the points $(0, -3)$ and $(3, 3)$, as shown in the graph below.

Now test whether the point $(0, 0)$ satisfies the inequality.

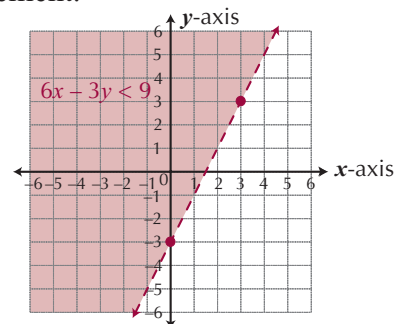
Substitute $x = 0$ and $y = 0$ into the inequality.

$$6x - 3y < 9$$

$$0 - 0 < 9$$

$$0 < 9 \text{ — This is a true statement.}$$

Therefore $(0, 0)$ lies in the region $6x - 3y < 9$ — so shade the region containing $(0, 0)$.



Check it out:

$(0, 0)$ has been used as a test point to make the algebra easy, but you can use any point — as long as it doesn't lie on the line.

Example 4

Graph the solution set of $y + 2x > 4$.

Solution

First plot the graph of the corresponding equation (the border line).

Table of values for sketching the line:

x	y	(x, y)
0	$y = -2x + 4 = -2(0) + 4 = 4$	(0, 4)
3	$y = -2x + 4 = -2(3) + 4 = -2$	(3, -2)

So the border line goes through the points (0, 4) and (3, -2), as shown below.

Test whether the point (0, 0) satisfies the inequality.

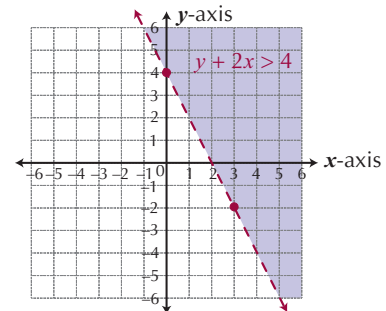
Substitute $x = 0$ and $y = 0$ into the inequality.

$$y + 2x > 4$$

$$0 + 0 > 4$$

$$0 > 4 \text{ — This is a false statement.}$$

Therefore (0, 0) does not lie in the region $y + 2x > 4$ — so shade the region that does not contain (0, 0).

**✓ Guided Practice**

In each of Exercises 3–8, use a set of axes spanning from -6 to 6 on the x - and y -axes. For each Exercise, shade the region defined by the inequality.

3. $y > 0.5x + 2$

4. $y + 2x < 0$

5. $y + x > -2$

6. $4x + 3y < 12$

7. $-2y + 3x > 6$

8. $y < -x + 3$

In Exercises 9–14, show whether the given point is a solution of $-5x + 2y > -8$.

9. (0, 0)

10. (6, -3)

11. (-3, 9)

12. (2, 1)

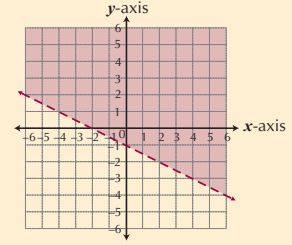
13. (39, -36)

14. (-15, 13)

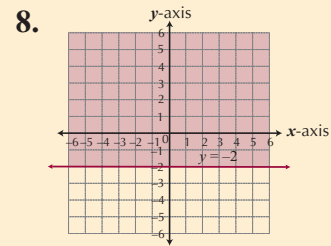
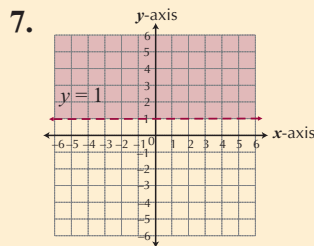
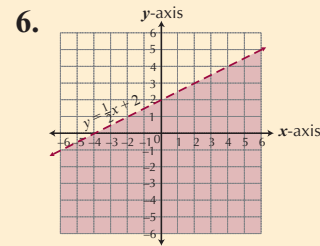
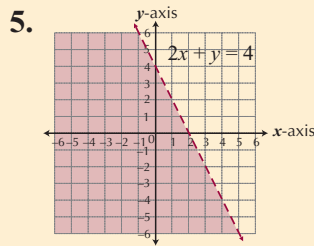
✓ Independent Practice

In Exercises 1–4, use the graph opposite to determine if the given point is in the solution set.

1. $(0, 0)$
2. $(1, 2)$
3. $(3, -2)$
4. $(-2, -2)$



In Exercises 5–8, state the inequality that defines the shaded region on each of the graphs.



9. Show whether $(-2, -1)$ is a solution of $2x - 5y \leq 10$.
10. Show whether $(-3, 5)$ is a solution of $4x + y < 5$.
11. Show whether $(2, 4)$ is a solution of $x + 6y < 0$.
12. Show whether $(0, -3)$ is a solution of $y \geq -6x + 5$.

Graph the solution set in Exercises 13–22.

13. $y < \frac{3}{4}x + 6$

14. $y < \frac{4}{5}x + 4$

15. $y < -\frac{2}{5}x - 2$

16. $y < 1$

17. $x > 0$

18. $x - 4y > 8$

19. $x + 2y > 8$

20. $4x + 3y < -12$

21. $4x - 6y > 24$

22. $5x + 8y < 24$

Round Up

Well, that was quite a long Topic, with lots of graphs. *Inequality graphs* aren't easy, so it's always a good idea to check whether you've *shaded the correct part* by seeing whether a *test point* satisfies the original inequality.

Borders of Regions

California Standards:

6.0: Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequality (e.g., they sketch the region defined by $2x + 6y < 4$).

What it means for you:

You'll learn how to show the different types of inequality on a graph.

Key words:

- inequality
- strict inequality
- border
- region

In Topic 4.5.1 you were dealing with regions defined by *strict inequalities* — the ones involving a $<$ or $>$ sign. This Topic shows you how to graph inequalities involving \leq and \geq signs too.

Regions Can Have Different Types of Borders

The region defined by a **strict** inequality **doesn't** include points on the **border line**, and you draw the border line as a **dashed** line.

For example, the region defined by $y > -x + 3$ doesn't include any points on the line $y = -x + 3$.

Regions defined by inequalities involving a \leq or \geq sign **do** include points on the **border line**. In this case, you draw the border line as a **solid** line.

For example, the region defined by $y \geq -x + 3$ includes all the points on the line $y = -x + 3$.

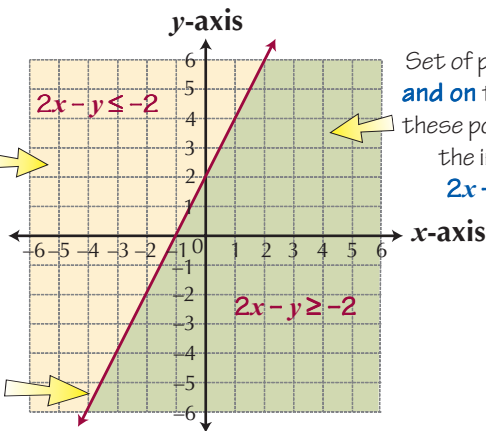
Example 1

Graph $2x - y = -2$ and show the three regions of the plane that include all the points on this line.

Solution

Set of points **above and on** the line — all these points satisfy the inequality $2x - y \leq -2$.

Set of points **on** the line — all these points satisfy the equation $2x - y = -2$ and both the inequalities.



Set of points **below and on** the line — all these points satisfy the inequality $2x - y \geq -2$.

Don't forget:

Remember, \leq means "less than or equal to," \geq means "greater than or equal to."

Sketching Regions Inclusive of the Border Line

The **method** for sketching the region is the same as the method in Topic 4.5.1, except now there's an extra step for showing the border type — using a **dashed** line, or a **solid** line.

Example 2

Sketch the region of the coordinate plane defined by $y \leq -x - 5$.

Solution

First plot the border line. The border line equation is $y = -x - 5$.
Table of values for sketching the line:

x	y	(x, y)
0	$y = -x - 5 = -0 - 5 = -5$	$(0, -5)$
-3	$y = -x - 5 = 3 - 5 = -2$	$(-3, -2)$

So the border line goes through the points $(0, -5)$ and $(-3, -2)$, as shown in the graph below.

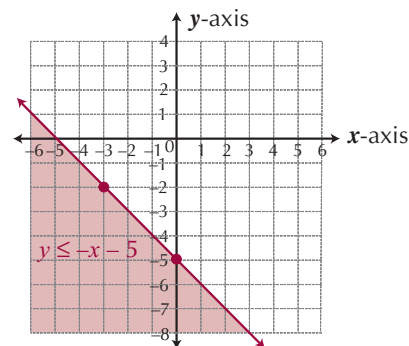
Identify the **border type**: the border line is **solid**, since the sign is \leq .

Test whether the point $(0, 0)$ satisfies the inequality — substitute $x = 0$ and $y = 0$ into the inequality.

$$y \leq -x - 5 \Rightarrow 0 \leq -0 - 5$$

$$0 \leq -5 \text{ — this is a false statement.}$$

Therefore $(0, 0)$ doesn't lie in the region $y \leq -x - 5$ — so shade the region that doesn't contain $(0, 0)$.



✓ Guided Practice

In Exercises 1–4, show whether the given point is in the solution set of $-2x + 3y \leq -15$.

1. $(0, 0)$ 2. $(6, -1)$ 3. $(4, -7)$ 4. $(-3, -5)$

In each of Exercises 5–8 use a set of axes spanning from -6 to 6 on the x - and y -axes, and shade the region defined by the inequality.

5. $4x + 3y \leq 9$ 6. $y \leq 2x + 3$
7. $-2y \geq x$ 8. $2x + y \geq 4$

More Examples of Graphing Regions

Graphing regions isn't always straightforward, so here are a couple more examples and some more practice exercises.

Example 3

Graph the solution set of $4y - 3x \geq 12$.

Solution

First form the border-line equation:

$$4y - 3x = 12$$

$$4y = 3x + 12$$

$$y = \frac{3}{4}x + 3$$

Table of values for sketching the line:

x	y	(x, y)
0	$y = \frac{3}{4}x + 3 = \frac{3}{4}(0) + 3 = 3$	(0, 3)
4	$y = \frac{3}{4}x + 3 = \frac{3}{4}(4) + 3 = 6$	(4, 6)

So the border line goes through the points (0, 3) and (4, 6).

The border line is **solid**, since the sign is \geq .

Test whether the point (0, 0) satisfies the inequality.

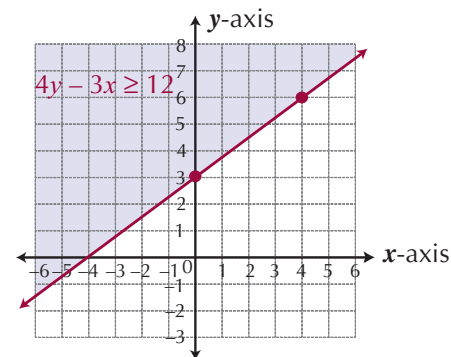
Substitute $x = 0$ and $y = 0$ into the inequality.

$$4y - 3x \geq 12$$

$$4(0) - 3(0) \geq 12$$

$$0 \geq 12 \text{ — This is a false statement.}$$

Therefore (0, 0) doesn't lie in the region $4y - 3x \geq 12$ — so shade the region that doesn't contain (0, 0).



Check it out:

After you've sketched a region, it can be a good idea to test another point to make sure you've got it right. Pick a point in your shaded region and make sure the inequality is satisfied there as well.

Example 4

Sketch the region of the coordinate plane defined by $2y < 2x + 6$.

Solution

First form the border-line equation:

$$2y = 2x + 6$$

$$y = x + 3$$

Table of values for sketching the line:

x	y	(x, y)
0	$y = x + 3 = 0 + 3 = 3$	(0, 3)
3	$y = x + 3 = 3 + 3 = 6$	(3, 6)

So the border line goes through the points (0, 3) and (3, 6).

The border line is **dashed**, since the sign is $<$.

Test whether the point (0, 0) satisfies the inequality.

Substitute $x = 0$ and $y = 0$ into the inequality.

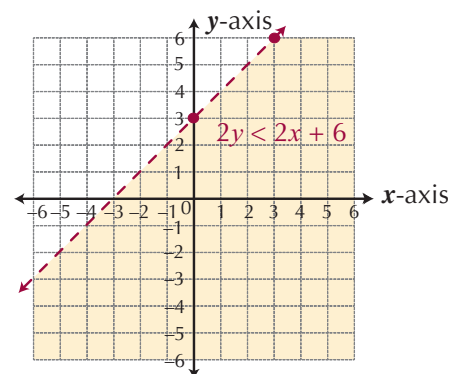
$$2y < 2x + 6$$

$$2(0) < 2(0) + 6$$

$$0 < 6 \text{ — This is a true}$$

statement.

Therefore (0, 0) lies in the region $2y < 2x + 6$ — so shade the region containing (0, 0).

**✓ Guided Practice**

In each of Exercises 9–18, use a set of axes spanning from -6 to 6 on the x - and y -axes. For each exercise, shade the region defined by the inequality.

9. $y \leq x$

10. $y - 2 < 0$

11. $x + 2 > 0$

12. $x - 4 \leq 0$

13. $y + 3 \leq 0$

14. $y \geq -3x$

15. $y \leq 2x - 5$

16. $x + 4y < 4$

17. $y > 2x + 6$

18. $4x - 3y \geq 8$

19. Show whether (5, 4) is in the solution set of $4x - 3y \leq 8$.

20. Show whether $(-4, 2)$ is in the solution set of $2x + y > -6$.

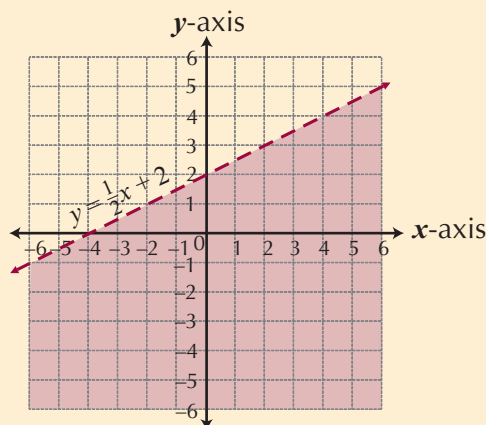
✓ Independent Practice

In Exercises 1–4, show whether the given point is in the solution set of the given line.

1. $(-1, 3)$; $2x + y < 1$
2. $(-2, -8)$; $x - y \geq 6$
3. $(0.5, 0.25)$; $x + 2y \leq 1$
4. $(1, -5)$; $x + 2y > -9$

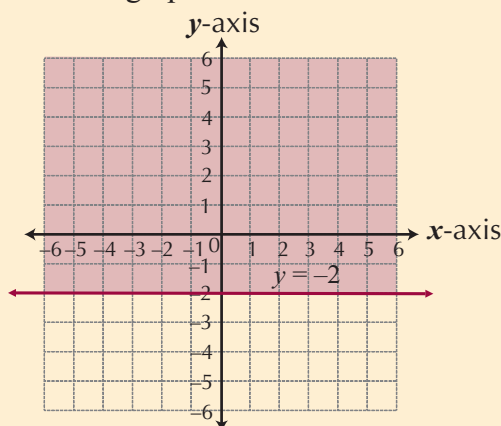
In Exercises 5–6, determine if the given point is in the solution set shown by the shaded region of the graph.

5. $(4, 6)$
6. $(-1, 1)$



In Exercises 7–8, determine if the given point is in the solution set shown by the shaded region of the graph.

7. $(3, -2)$
8. $(-2, 3)$



Graph the solution set in Exercises 9–20.

9. $x > 4$
10. $y \leq -2$
11. $x - 3y \leq 9$
12. $x + y < 5$
13. $x + y \geq -1$
14. $y \geq 2x - 7$
15. $x + 2y < 2$
16. $y \leq 2x + 2$
17. $y \geq 3x$
18. $y > -2x + 1$
19. $x + y \geq 3$
20. $2x + y \leq -3$

Round Up

Remember — graphs of inequalities including $<$ and $>$ signs will always have a dashed line, and graphs including \leq and \geq signs will always have a solid line.

Regions Defined by Two Inequalities

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. **Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.**

What it means for you:

You'll graph two inequalities to show the solution set that satisfies both inequalities.

Key words:

- system of linear inequalities
- region
- point-slope formula

The idea of graphing two different inequalities on one graph is really not as hard as it sounds. When you've finished, your graph will show the region where all the points satisfy both inequalities.

Regions Defined by More Than One Linear Inequality

A **system of linear inequalities** is made up of **two or more** linear inequalities that contain the same variables.

For example, $3x + 2y > 6$ and $4x - y < 5$ are linear inequalities both containing the variables x and y .

An **ordered pair** (x, y) is a **solution** of a **system of linear inequalities** if it is a solution of each of the inequalities in the system.

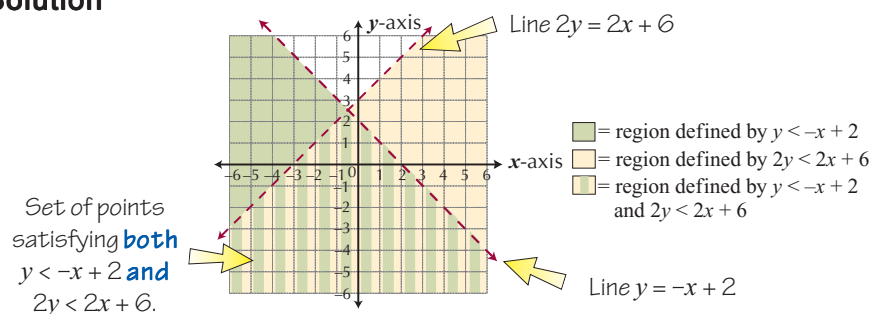
For example, $(1, -2)$ is a solution of the system of inequalities $y < -x + 2$ and $2y < 2x + 6$.

The **graph** of two linear inequalities is the region consisting of all the points satisfying **both** inequalities.

Example 1

Sketch the region satisfying both $y < -x + 2$ and $2y < 2x + 6$.

Solution



Sketching Regions Defined by Two Linear Inequalities

To **sketch** the region defined by **two linear inequalities**, shade the regions defined by each inequality. The **region defined by both inequalities** is the area where your shading **overlaps**.

Check it out:

The problem in Example 2 is just another way of asking you to sketch the region defined by $5y + 3x \geq -25$ and $y - x \leq -5$.

Example 2

Graph the solution set satisfying $5y + 3x \geq -25$ and $y - x \leq -5$.

Solution

First line:

$$5y + 3x = -25 \Rightarrow 5y = -3x - 25 \Rightarrow y = -\frac{3}{5}x - 5$$

x	y	(x, y)
0	$y = -\frac{3}{5}x - 5 = -\frac{3}{5}(0) - 5 = -5$	(0, -5)
5	$y = -\frac{3}{5}x - 5 = -\frac{3}{5}(5) - 5 = -8$	(5, -8)

The border line $y = -\frac{3}{5}x - 5$ goes through the points (0, -5) and (5, -8), and is a solid line.

Second line:

$$y - x = -5 \Rightarrow y = x - 5$$

x	y	(x, y)
0	$y = x - 5 = 0 - 5 = -5$	(0, -5)
3	$y = x - 5 = 3 - 5 = -2$	(3, -2)

The border line $y = x - 5$ goes through the points (0, -5) and (3, -2), and is a solid line.

Test whether the point (0, 0) satisfies each inequality:

$$\begin{aligned} 5y + 3x &\geq -25 \\ 5(0) + 3(0) &\geq -25 \\ 0 &\geq -25 \end{aligned}$$

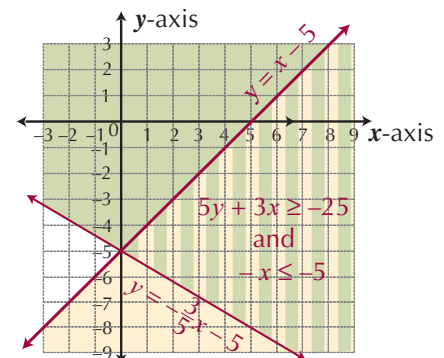
This is a true statement, so (0, 0) lies in the region $5y + 3x \geq -25$.

Shade the region above $y = -\frac{3}{5}x - 5$.

$$\begin{aligned} y - x &\leq -5 \\ 0 - 0 &\leq -5 \\ 0 &\leq -5 \end{aligned}$$

This is a false statement, so (0, 0) doesn't lie in the region $y - x \leq -5$. Shade the region below $y = x - 5$.

The required region is the area where the shading overlaps.



Don't forget:

You need to graph both lines on the same axes.

✓ Guided Practice

In each of Exercises 1–2, use a set of axes spanning from -5 to 7 on the x -axis and -6 to 6 on the y -axis. For each exercise, shade the region containing all solution points for both inequalities.

1. $2x + 3y < 6$ and $y - 2x < 2$ 2. $y - x \geq 4$ and $2x + y \leq 5$

In each of Exercises 3–6, use a set of axes spanning from -6 to 6 on the x - and y -axes. For each exercise, shade the region containing all solution points for both inequalities.

3. $y < -x + 4$ and $y < x$ 4. $y < x + 2$ and $y > -2x + 5$
5. $y < x$ and $y < -x - 2$ 6. $y \leq 0.5x + 3$ and $y \leq 2x + 2$

Identifying the Inequalities Defining a Region

To **identify the inequalities** defining a region, you first need to establish the **border line equations**. Then **examine a point in the region** to identify the inequalities. Here's the **method**:

- **Find the equations** — use the **point-slope formula** to find the **equations** of each of the border lines.
- **Identify the signs** — choose a point in the region (but **not** on a line) and **substitute** its coordinates into each equation. Since the point does not lie on either of the lines, you will have two **false** statements. Replace the “=” in each equation with a “<” or “>” sign to make the statements true. If the line is **solid**, use “≤” or “≥.”
- **Write the inequalities** which define the region.

Example 3

Find the inequalities whose simultaneous solution set is the shaded region shown on the right.

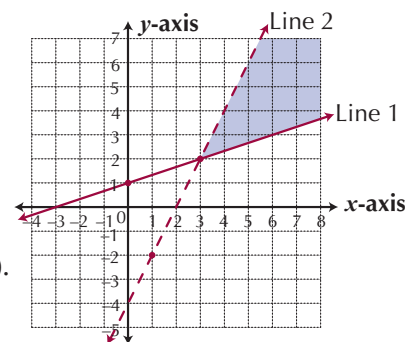
Solution

First line:

Two points on this line are $(0, 1)$ and $(3, 2)$.

$$m_1 = \frac{2-1}{3-0} = \frac{1}{3}$$

$$y - y_1 = m(x - x_1) \Rightarrow y - 2 = \frac{1}{3}(x - 3) \Rightarrow y - 2 = \frac{1}{3}x - 1 \Rightarrow y = \frac{1}{3}x + 1$$



Example 3 continued

Second line:

Two points on this line are (3, 2) and (1, -2).

$$m_1 = \frac{-2 - 2}{1 - 3} = \frac{-4}{-2} = 2$$

$$\begin{aligned}y - y_1 &= m(x - x_1) \Rightarrow y - 2 = 2(x - 3) \\ &\Rightarrow y - 2 = 2x - 6 \\ &\Rightarrow y = 2x - 4\end{aligned}$$

So the equations of the two border lines are $y = \frac{1}{3}x + 1$ and $y = 2x - 4$.

Choose a point in the shaded region, for example, (5, 4). Substitute this point into the two equations.

Equation for line 1:

$$y = \frac{1}{3}x + 1 \Rightarrow 4 = \frac{5}{3} + 1 \Rightarrow 4 = \frac{8}{3} \text{ — this is a false statement.}$$

$4 > \frac{8}{3}$, so a $>$ sign is needed to make it true.

However, the line is solid so the sign should be \geq .

So the first inequality is $y \geq \frac{1}{3}x + 1$.

Equation for line 2:

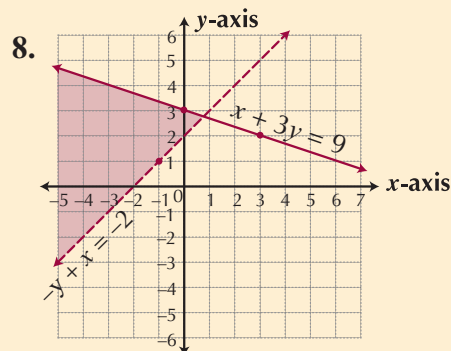
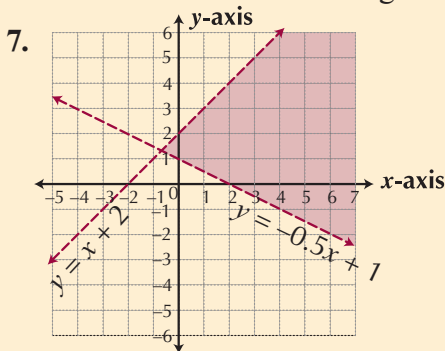
$$y = 2x - 4 \Rightarrow 4 = 10 - 4 \Rightarrow 4 = 6 \text{ — this is a false statement.}$$

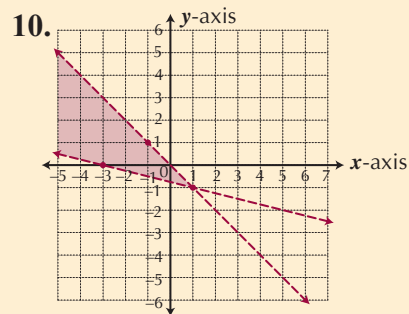
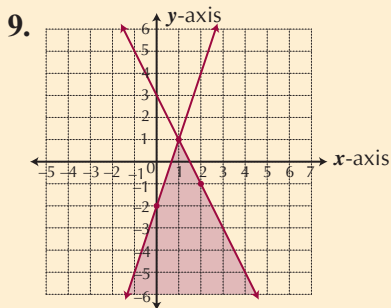
$4 < 6$, so a $<$ sign is needed to make it true. The line is dashed, so the $<$ sign is correct. So the second inequality is $y < 2x - 4$.

Therefore the inequalities defining the shaded region are $y \geq \frac{1}{3}x + 1$ and $y < 2x - 4$.

✓ Guided Practice

In Exercises 7–10, find inequalities whose simultaneous solution defines each of the shaded regions.

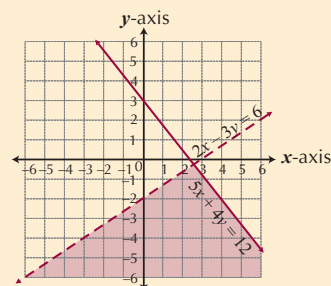




✓ Independent Practice

In Exercises 1–4, use the graph opposite to determine if the given point is included in the solution set.

1. $(0, 0)$
2. $(2, 6)$
3. $(6, 2)$
4. $(4, -2)$



5. Determine whether the point $(-4, -3)$ lies in the solution region of both $3x - 4y \leq 2$ and $x - 2y \geq 1$.
6. Determine whether the point $(0, 0)$ lies in the solution region of both $3x - 4y \leq 2$ and $x - 2y \geq 1$.
7. Determine whether the point $(-3, 1)$ lies in the solution region of both $3x - 4y \leq 2$ and $x - 2y \geq 1$.
8. On axes spanning from -3 to 9 on the x -axis and -6 to 6 on the y -axis, graph the solution set that satisfies both the inequalities $y + 2x \geq 4$ and $y \geq 1.5x - 2$.
9. On axes spanning from -3 to 9 on the x -axis and -8 to 4 on the y -axis, graph the solution set that satisfies both the inequalities $y > x - 6$ and $y > -x + 2$.

In each of Exercises 10–11, show on axes spanning from -5 to 7 on the x -axis and -6 to 6 on the y -axis the region defined by the set of inequalities:

10. $4y - 3x < 12$, $2x + y < 0$, and $y \geq 1$
11. $3x + 5y > 10$, $x - y < 2$, and $y \leq 2$
12. The points $(2, 6.6)$ and $(7, 8.1)$ lie on a line bounding a region. Find the equation of the line.
13. The region in Exercise 12 is also bounded by the lines $y = 0.3x$ and $y = |1.5x - 6|$. If the points $(2, 3)$, $(5, 1.5)$, and $(4, 4)$ all lie within the region, draw and shade the region on axes spanning from -2 to 12 on the x - and y -axes.

Round Up

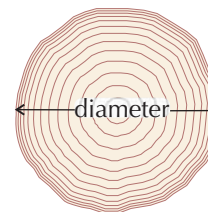
When you're graphing a system of linear inequalities, don't forget that you still have to pay attention to whether the lines should be solid or dashed.

Chapter 4 Investigation

Tree Growth

This Chapter was all about turning raw *data* into *graphs* that show you *patterns* in the data.

The loss of forest areas (**deforestation**) is a huge **environmental problem**. Many countries are cutting down trees at a faster rate than they can be replaced. If you look at a cross section of a tree, you'll notice a pattern of **rings**. Each ring is a layer of wood that took **one year** to grow.



26 pine trees have been felled.

The **number of rings** on each stump and the **diameter** are recorded below.

Number of rings (age)	23	69	15	4	46	19	20	53	68	32	27	32	32
Diameter (cm)	30.6	62.1	25.5	11.2	41.4	9.2	20.7	42.0	47.1	11.1	20.4	27.0	31.8

Number of rings (age)	50	37	11	48	8	43	42	63	16	53	64	57	58
Diameter (cm)	49.3	40.0	13.7	14.3	4.8	30.9	33.4	53.5	4.8	33.7	35.0	45.5	52.5

- Draw a **scatter diagram** of the data. Put “Age (years)” on the *x*-axis and “Diameter (cm)” on the *y*-axis.
- Describe the **correlation** between the **age** of the tree and the **diameter** of its trunk. Why do you think that the correlation is not perfect?
- Draw a **best-fit line** for the data. A best-fit line is one that passes as close to as many of the points as possible. About half the points should be on each side of the line. Would you expect this line to pass through the **origin**? Why?
- What is the **slope** of the best-fit line? What does the slope of your graph represent?
- Find the **equation** of the best-fit line.
- a) How **old** would you expect a tree to be if its diameter is 23 cm?
b) What would you expect the **diameter** of a 6-year-old tree to be?

Extension

It's difficult to measure the **diameter** of a living tree. It's much easier to measure the **circumference** of the tree instead.

- 1) Write an **equation** to link a tree's circumference with its likely age. How old is a tree likely to be if its circumference is 157 cm?
- 2) What is the **average annual increase** in a tree's circumference?

Open-ended Extension (For this investigation you will need a selection of books of different types.)

- 1) Record the **number of pages** and the **thickness** of each book (excluding the cover). Produce a **scatter diagram** of the data and draw a **best-fit line**. Find the **slope** and **equation** of your line. What does the slope mean? Measure the **thickness** of some different books. Use your equation to **predict** the number of pages. How accurate are your predictions? Why do you think this is?
- 2) Think of two sets of data that you suspect may have a **linear relationship**. You should choose data sets that you can easily collect. Investigate the relationship using a **scatter diagram**.

Round Up

You can work out patterns in any *collection of data*, as long as the data is all about the same thing.

Chapter 5

Systems of Equations

Section 5.1	Systems of Equations	229
Section 5.2	The Elimination Method	244
Section 5.3	Applications of Systems of Equations ..	249
Investigation	Breaking Even in an Egg Business	261

Section 5.1

The Graphing Method

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

What it means for you:

You'll solve systems of linear equations by graphing the lines and working out where they intersect.

Key words:

- system of linear equations
- simultaneous equations

In Section 4.5 you graphed two inequalities to find the region of points that satisfied both inequalities. Plotting two linear equations on a graph involves fewer steps, and it means you can show the solution to both equations graphically.

Systems of Linear Equations

A **system of linear equations** consists of **two or more** linear equations in the **same variables**. For example, $3x + 2y = 7$ and $x - 3y = -5$ form a **system of linear equations in two variables** — x and y .

The **solution** of a **system of linear equations in two variables** is a **pair of values** like x and y , or (x, y) , that satisfies each of the equations in the system. For example, $x = 1$, $y = 2$ or $(1, 2)$ is the **solution** of the system of equations $3x + 2y = 7$ and $x - 3y = -5$, since it satisfies both equations.

Equations in a system are often called **simultaneous** equations because any solution has to satisfy the equations simultaneously (at the same time). The equations **can't** be solved independently of one another.

Solving Systems of Equations by Graphing

A **system of two linear equations** can be **solved graphically**, by graphing both equations in the same coordinate plane. **Every** point on the line of an equation is a solution of that equation.

The point at which the two lines cross lies on **both** lines and so is the **solution of both equations**.

Check it out:

The point of intersection is the point where the lines cross.

The **solution** of a system of linear equations in two variables is the **point of intersection** (x, y) of their graphs.

Example 1

Solve this system of equations by graphing: $2x - 3y = 7$
 $-2x + y = -1$

Check it out:

Use whichever method you find easiest to plot the graphs. The tables of values method's been used in Example 1.

Solution

Step 1: Graph both equations in the same coordinate plane.

Line of first equation:

$$2x - 3y = 7$$

$$3y = 2x - 7$$

$$y = \frac{2}{3}x - \frac{7}{3} \Rightarrow$$

x	y
2	-1
-1	-3

The line goes through the points (2, -1) and (-1, -3).

Line of second equation:

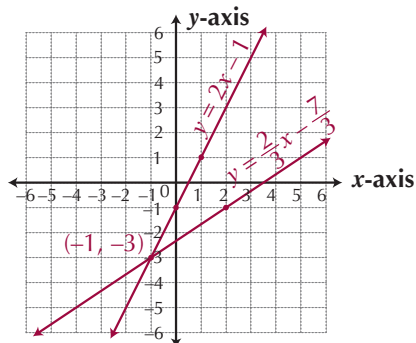
$$-2x + y = -1$$

$$y = 2x - 1 \Rightarrow$$

x	y
0	-1
1	1

The line goes through the points (0, -1) and (1, 1).

Now you can draw the graph:



Step 2: Read off the coordinates of the point of intersection. The point of intersection is (-1, -3).

Step 3: Check whether your coordinates give true statements when they are substituted into each equation.

$$2x - 3y = 7 \Rightarrow 2(-1) - 3(-3) = 7$$

$$\Rightarrow 7 = 7 \text{ — True statement}$$

$$-2x + y = -1 \Rightarrow -2(-1) + (-3) = -1$$

$$\Rightarrow -1 = -1 \text{ — True statement}$$

Therefore $x = -1, y = -3$ is the solution of the system of equations.

✓ Guided Practice

Solve each system of equations in Exercises 1–6 by graphing on x - and y -axes spanning from -6 to 6 .

1. $y + x = 2$ and $y = \frac{2}{3}x + 2$

2. $y + x = 3$ and $3y - x = 5$

3. $y = x - 3$ and $y + 2x = 3$

4. $y - \frac{3}{2}x = 1$ and $y + \frac{1}{2}x = -3$

5. $y - x = 3$ and $y + x = -1$

6. $2y - x = -6$ and $y + \frac{1}{2}x = -3$

✓ Independent Practice

Solve each system of equations in Exercises 1–6 by graphing on x - and y -axes spanning from -6 to 6 .

1. $2x + y = 7$ and $y = x + 1$

2. $x + y = 0$ and $y = -2x$

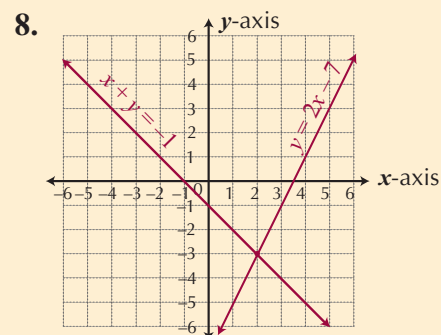
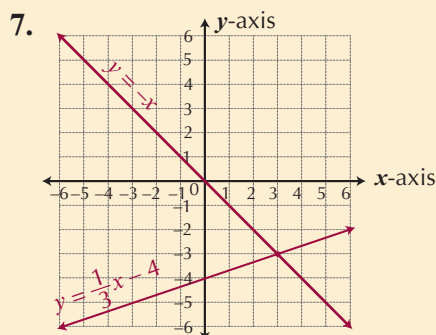
3. $y = -3$ and $x - y = 2$

4. $x - y = 4$ and $x + 4y = -1$

5. $2y + 4x = 4$ and $y = -x + 3$

6. $y = -x$ and $y = 4x$

Determine the solution to the systems of equations graphed in Exercises 7 and 8.



Solve each system of equations in Exercises 9–16 by graphing on x - and y -axes spanning from -6 to 6 .

9. $x - y = 6$ and $x + y = 0$

10. $y = 2x - 1$ and $x + y = 8$

11. $4x - 3y = 0$ and $4x + y = 16$

12. $x - y = 0$ and $x + y = 8$

13. $y = -x + 6$ and $x - y = -4$

14. $x - y = 1$ and $x + y = -3$

15. $x + y = 1$ and $x - 2y = 1$

16. $2x + y = -8$ and $3x + y = -13$

Round Up

There's something very satisfying about taking two long linear equations and coming up with just a *one-coordinate-pair solution*. You should always *substitute* your solution back into the original equations, to *check* that you've got the correct answer.

The Substitution Method

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

What it means for you:

You'll learn about the substitution method and use it to solve systems of linear equations.

Key words:

- substitution
- system of linear equations
- simultaneous equations

Check it out:

This is the same problem that you saw in Example 1 in the previous Topic — but this time it's been solved with the substitution method.

Graphing can work well if you have *integer solutions*, but it can be difficult to read off *fractional solutions*, and even integer solutions if the scale of your graph is small.

The Substitution Method Doesn't Involve Graphs

The **substitution method** involves a bit more algebra than graphing, but will generally give you more **accurate** solutions.

Substitution Method

- Take one of the equations and **solve for one of the variables** (x or y).
- **Substitute** the expression for the variable into the other equation. This gives an equation with just **one variable**. Solve this equation to find the value of the variable.
- **Substitute** this value into one of the earlier equations and solve it to find the value of the second variable.

Example 1

Solve this system of equations using substitution: $2x - 3y = 7$ (Equation 1)
 $-2x + y = -1$ (Equation 2)

Solution

Step 1: Rearrange one equation so that one of the variables is expressed in terms of the other. In this case, it's easiest to solve for y in Equation 2 because it has a coefficient of 1:

$$-2x + y = -1 \Rightarrow y = 2x - 1 \quad (\text{Equation 3})$$

Now you have y expressed in terms of x .

Step 2: Substitute $2x - 1$ for y in Equation 1. Then solve to find the value of x .

$$\begin{aligned} 2x - 3y &= 7 \\ 2x - 3(2x - 1) &= 7 \\ 2x - 6x + 3 &= 7 \\ -4x + 3 &= 7 \\ -4x &= 4 \\ x &= -1 \end{aligned}$$

Example 1 continued

Step 3: Substitute -1 for x into an equation to find y .

Equation 3 is the best one to use here as y is already isolated — so you don't have to do any rearranging.

$$y = 2x - 1 \Rightarrow y = 2(-1) - 1 \Rightarrow y = -3$$

Therefore $x = -1, y = -3$ is the solution of the system of equations.

It's a good idea to check that the solution is correct by substituting it into the original equations.

$$\begin{aligned} 2x - 3y = 7 &\Rightarrow 2(-1) - 3(-3) = 7 \\ &\Rightarrow -2 + 9 = 7 \\ &\Rightarrow 7 = 7 \text{ — True statement} \end{aligned}$$

$$\begin{aligned} -2x + y = -1 &\Rightarrow -2(-1) + (-3) = -1 \\ &\Rightarrow 2 - 3 = -1 \\ &\Rightarrow -1 = -1 \text{ — True statement} \end{aligned}$$

The solution makes both of the original equations true statements, so it must be correct.

✓ Guided Practice

Solve each system of equations in Exercises 1–6 by the substitution method.

- $y = 2x - 3$ and $-3y + 2x = -15$
- $x + 2y = 7$ and $3x - 2y = 5$
- $a + b = 2$ and $5a - 2b = -4$
- $w + z = 13$ and $w - 2z = 4$
- $y + 2x = 5$ and $2x - 3y = 1$
- $m + 6n = 25$ and $n - 3m = -18$

✓ Independent Practice

Solve by the substitution method:

- $y = 3x$ and $x + 21 = -2y$
- $x + y = 5$ and $5x + 2y = 16$
- $7x - 4y = 27$ and $-x + 4y = 3$
- $9y - 6x = 0$ and $13x + 9 = 24y$

Solve each system of three equations by the substitution method:

- $3x + y = 10, 4x - z = 7, 7y + 2z = 17$
- $\frac{1}{2}a + \frac{1}{3}b + \frac{1}{4}c = 7, 2a + b - 2c = -11, -4a - 4b + c = -32$

Round Up

With the substitution method, it doesn't matter which equation you choose to rearrange, or which variable you choose to solve for — you will get the *same answer* as long as you follow the steps correctly and don't make any mistakes along the way.

The important thing is to keep the algebra as *simple* as you possibly can.

Comparing the Graphing and Substitution Methods

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

What it means for you:

You'll show that the graphing and substitution methods both give the same solution to a system of linear equations.

Key words:

- substitution
- system of linear equations
- simultaneous equations

There's nothing new in this Topic — you'll just see that the graphing and substitution methods always give the same answer.

An Example Using the Graphing Method

Example 1

Solve this system of equations by graphing: $2y + x = 8$
 $2y + 2x = 10$

Solution

Step 1: Graph both equations in the same coordinate plane.

Line of first equation:

$$2y + x = 8 \Rightarrow 2y = -x + 8$$

$$\Rightarrow y = -\frac{1}{2}x + 4$$

x	y
0	4
2	3

The line goes through the points (0, 4) and (2, 3).

Line of second equation:

$$2y + 2x = 10 \Rightarrow 2y = -2x + 10$$

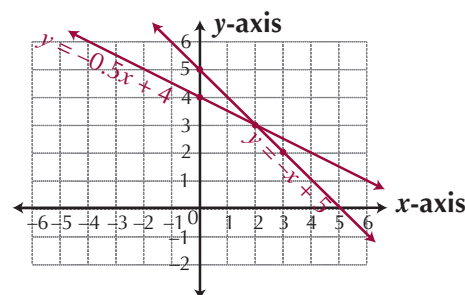
$$\Rightarrow y = -x + 5$$

x	y
0	5
3	2

The line goes through the points (0, 5) and (3, 2).

Step 2: Read off the coordinates of the point of intersection.

The point of intersection is (2, 3).



Step 3: Check whether your coordinates give true statements when substituted into each of the equations.

$$2y + x = 8 \Rightarrow 2(3) + 2 = 8$$

$$\Rightarrow 8 = 8 \text{ — True statement}$$

$$2y + 2x = 10 \Rightarrow 2(3) + 2(2) = 10$$

$$\Rightarrow 10 = 10 \text{ — True statement}$$

Therefore $x = 2, y = 3$ is the solution of the system of equations.

The Same Example Using the Substitution Method

Example 2

Solve this system of equations using substitution: $2y + x = 8$ (Equation 1)
 $2y + 2x = 10$ (Equation 2)

Solution

Step 1: Rearrange one equation so that one of the variables is expressed in terms of the other. In this case, it's easiest to solve for x in Equation 1 because it has a coefficient of 1:

$$2y + x = 8 \Rightarrow x = -2y + 8 \quad (\text{Equation 3})$$

Now you have x expressed in terms of y .

Step 2: Substitute $-2y + 8$ for x in Equation 2. Then solve to find the value of y .

$$\begin{aligned} 2y + 2x &= 10 \\ 2y + 2(-2y + 8) &= 10 \\ 2y - 4y + 16 &= 10 \\ -2y &= -6 \\ y &= 3 \end{aligned}$$

Step 3: Substitute 3 for y in an equation to find x . Equation 3 is the best one to use here because x is already isolated — so you don't have to do any rearranging.

$$\begin{aligned} x &= -2y + 8 \\ x &= -2(3) + 8 \\ x &= 2 \end{aligned}$$

So $x = 2, y = 3$, or $(2, 3)$, is the solution of the system of equations.

Check by substituting the solution in the original equations.

$$\begin{aligned} 2y + x &= 8 \\ 2(3) + 2 &= 8 \\ 8 &= 8 \text{ — True statement} \end{aligned}$$

$$\begin{aligned} 2y + 2x &= 10 \\ 2(3) + 2(2) &= 10 \\ 10 &= 10 \text{ — True statement} \end{aligned}$$

The solution makes both of the original equations true statements, so it must be correct.

Check it out:

The substitution method gives the same answer as the graphing method used in Example 1.

✓ Guided Practice

Solve each system of equations in Exercises 1–4 by graphing on x - and y -axes spanning from -6 to 6 .

1. $y = 5$ and $x + y = 9$

2. $x = 4$ and $x + y = 6$

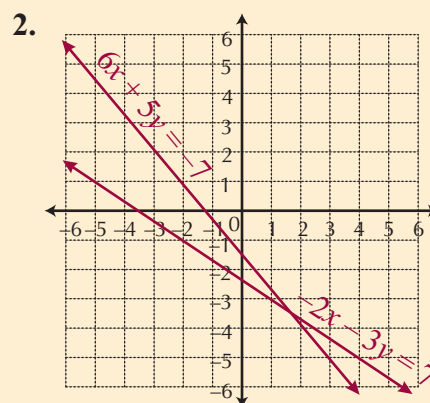
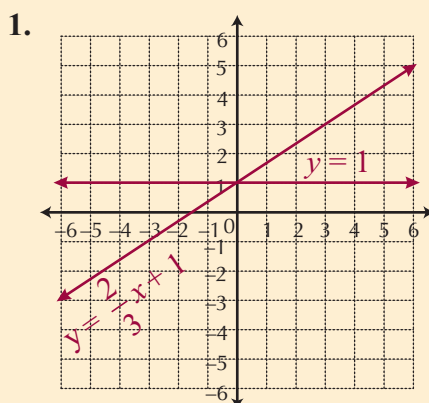
3. $4x + 4y = 12$ and $2x + y = 5$

4. $5x + y = 15$ and $y = \frac{1}{3}x - 1$

5. Solve the systems of equations in Exercises 1–4 using the substitution method.

✓ Independent Practice

In Exercises 1–2, say whether it is possible to find the exact solutions of each system of equations using their graphs.



Solve Exercises 3–6 by graphing.

3. $y = 2$ and $3x - y = 10$

4. $x - y = 0$ and $x + y = 6$

5. $x - y = 1$ and $2x + y = -1$

6. $x - 2y = 4$ and $y = -x + 4$

7. Solve the systems of equations in Exercises 3–6 using the substitution method.

Solve Exercises 8–13 by substitution.

8. $2x - y = 8$ and $y = 4$

9. $x + y = 0$ and $y = -3x$

10. $y = 2x - 1$ and $x + y = 5$

11. $y = 3x$ and $2x + 3y = 44$

12. $6x + y = -2$ and $4x - 3y = 17$

13. $4x - 5y = 0$ and $4x - 3y = 8$

Round Up

The *graphing method* works well with *simple equations with integer solutions*, but you need to draw the graphs very carefully — if your graphs aren't accurate enough, you'll get the wrong solution. If the equations look complicated, the *substitution method* is likely to give more *accurate results*.

Inconsistent Systems

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

What it means for you:

You'll learn how to spot systems of equations that have no solutions.

Key words:

- inconsistent
- point of intersection
- parallel
- substitution
- system of linear equations

So far, there's been one solution to each system of linear equations — but it doesn't always work out like that.

Systems of Linear Equations

The systems so far in this Section have all had **one solution**. These are called **independent systems**.

Some systems of two linear equations have **no solutions** — in other words, there's **no ordered pair (x, y)** that satisfies **both** of the equations in the system.

A system of equations with **no solutions** is called an **inconsistent** system.

Inconsistent System Lines Don't Intersect

Example 1

Solve this system of equations by graphing:

$$\begin{aligned} y - x &= 1 \\ y &= x - 3 \end{aligned}$$

Solution

Step 1: Graph both equations in the same coordinate plane.

Line of first equation:

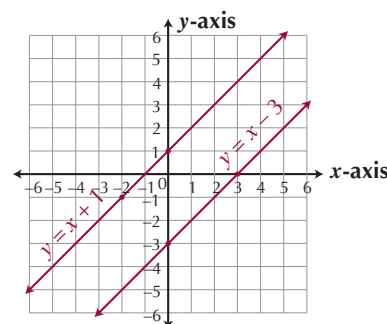
$$\begin{aligned} y - x &= 1 \\ y &= x + 1 \end{aligned} \Rightarrow \begin{array}{|c|c|} \hline x & y \\ \hline 0 & 1 \\ \hline -2 & -1 \\ \hline \end{array}$$

The line goes through $(0, 1)$ and $(-2, -1)$.

Line of second equation:

$$y = x - 3 \Rightarrow \begin{array}{|c|c|} \hline x & y \\ \hline 0 & -3 \\ \hline 3 & 0 \\ \hline \end{array}$$

The line goes through $(0, -3)$ and $(3, 0)$.



Step 2: Read off the coordinates of the point of intersection.

The lines are parallel, so there's **no point of intersection**. So the system of equations has no solutions — **the system is inconsistent**.

The **lines** of equations in an **inconsistent** system are **parallel**.

Check it out:

Equations with identical slopes, but different intercepts, result in parallel lines. So you can tell if the system is inconsistent without even drawing the graphs, by rearranging the equations into the slope-intercept form.

The Substitution Method Gives a False Statement

Here's the same problem you saw in Example 1, but this time using the substitution method.

Example 2

Solve this system of equations by substitution: $y - x = 1$ (Equation 1)

$y = x - 3$ (Equation 2)

Solution

Step 1: Rearrange one equation so that one of the variables is expressed in terms of the other. Equation 2 already expresses y in terms of x :

$$y = x - 3$$

Step 2: Substitute $x - 3$ for y in Equation 1. Then solve to find x .

$$y - x = 1$$
$$(x - 3) - x = 1 \quad (\text{Equation 3})$$

$$x - 3 - x = 1$$

$-3 = 1$, which is a **false** statement.

Therefore Equation 3 doesn't hold for any value of x .

So the system of equations has no solutions
— the system is inconsistent.

Check it out

You can't substitute any value of x into Equation 3 and make a true statement — so there are no solutions.

Guided Practice

1. How many solutions does an independent system have?
2. How many solutions does an inconsistent system have?
3. Convert $y = 3x - 4$ and $-9x + 3y = 6$ to slope-intercept form.
4. Say why you can now tell that the equations from Exercise 3 must be inconsistent.
5. Use the substitution method to say whether $y = 3x - 4$ and $-9x + 3y = 7$ are inconsistent.

Independent Practice

In Exercises 1–4, graph each pair of equations and say whether they are independent or inconsistent.

1. $y = x - 2$ and $y - x = -6$

2. $2y + x = 5$ and $y = -x + 1$

3. $2x - y = -1$ and $y = 2x + 2$

4. $3x + 4y = 3$ and $y = -\frac{3}{4}x - 4$

Use the substitution method to show that each of the pairs of equations in Exercises 5–8 are inconsistent.

5. $x + y = -5$ and $x + y = 3$

6. $x + y = -6$ and $y = 4 - x$

7. $2x + y = -3$ and $2x + y = 1$

8. $x + 2y = -2$ and $2x = 10 - 4y$

Round Up

From this Topic you've seen that it doesn't matter whether you use the graphing or substitution method — if the system is **inconsistent**, you **won't get any solutions**.

Topic 5.1.5

Dependent Systems

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

What it means for you:

You'll learn how to spot systems of equations with infinitely many solutions.

Key words:

- dependent
- coincide
- infinite
- substitution
- system of linear equations

In Topic 5.1.4 you saw systems of equations with *no solution*. In this Topic you'll see systems that have *infinitely many solutions*.

A Dependent System Has Infinitely Many Solutions

Some systems of two linear equations have an **infinite number of solutions** — in other words, there are an infinite number of points (x, y) that satisfy **both** of the equations in the system.

Every solution of one equation is also a solution of the other equation.

A system of equations with an **infinite number of solutions** is said to be **dependent**.

Graphing a Dependent System — the Lines Coincide

Example 1

Solve this system of equations by graphing:

$$\begin{aligned}y + x &= 4 \\ 2y + 2x &= 8\end{aligned}$$

Solution

Step 1: Graph both equations in the same coordinate plane.

Line of first equation:

$$y + x = 4$$

$$y = -x + 4$$

x	y
0	4
2	2

The line goes through $(0, 4)$ and $(2, 2)$.

Line of second equation:

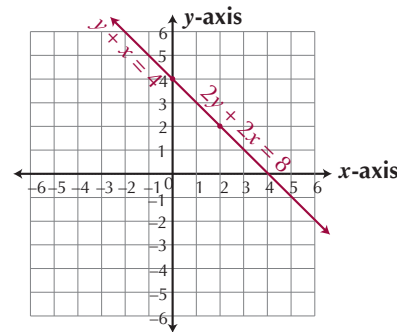
$$2y + 2x = 8$$

$$2y = -2x + 8$$

$$y = -x + 4$$

x	y
0	4
2	2

The line goes through $(0, 4)$ and $(2, 2)$.



Step 2: Read off the coordinates of the point of intersection.

The two equations represent the same line, so there are infinitely many points of intersection.

Therefore the system of equations has infinitely many solutions — the system is dependent.

Check it out:

Every point on the line $y + x = 4$ is on the line $2y + 2x = 8$.

The lines of equations in a **dependent** system **coincide**.

The Substitution Method Gives a True Statement

Here's the same problem you saw in Example 1, but this time using the substitution method.

Example 2

Solve this system of equations by substitution: $y + x = 4$ (Equation 1)
 $2y + 2x = 8$ (Equation 2)

Solution

Step 1: Rearrange one equation so that one of the variables is expressed in terms of the other. Here, it's easiest to solve for x or y in Equation 1.

$$\begin{aligned}y + x &= 4 \\y &= -x + 4 \quad (\text{Equation 3})\end{aligned}$$

Step 2: Substitute $-x + 4$ for y in Equation 2. Then solve to find x .

$$\begin{aligned}2y + 2x &= 8 \\2(-x + 4) + 2x &= 8 \quad (\text{Equation 4}) \\-2x + 8 + 2x &= 8 \\8 &= 8, \text{ which is a } \mathbf{true} \text{ statement.}\end{aligned}$$

Therefore Equation 4 holds for any value of x .

So the system of equations has infinitely many solutions — the system is dependent.

Check it out:

You can substitute any value of x into Equation 4 and make a true statement because x disappears. So there are infinitely many solutions.

Guided Practice

1. How many solutions does a dependent system have?

Say whether the systems in Exercises 2–5 are dependent.

2. $x - y = 5$ and $2x = 2y + 10$ 3. $2x + 2y = 2$ and $y = -x + 1$
4. $4y - 5x = 12$ and $y = \frac{5}{4}x + 3$ 5. $-2x + 7y = -14$ and $y = \frac{2}{7}x + 1$

Independent Practice

By graphing, say whether the systems in Exercises 1–4 are dependent.

1. $3x - y = -1$ and $-3x + y = 1$ 2. $y - 3x = -1$ and $-2y + 6x = 2$
3. $y = -2x + 6$ and $y = \frac{1}{2}x - 4$ 4. $4y + 8x = 8$ and $y + 2x = 2$

Use substitution to say whether the systems in Exercises 5–8 are dependent.

5. $x + y = 2$ and $2x + 2y = 4$ 6. $x + y = 57$ and $y = 6x + 120$
7. $y = -\frac{1}{2}x + \frac{7}{2}$ and $2y + x = 7$ 8. $y = -\frac{4}{3}x - 4$ and $3y + 4x = -12$

Round Up

It would be crazy to try to *list* the solutions to *dependent systems*, because there are *infinitely many*. You can actually rearrange the equations and show that they're really saying the same thing.

Topic 5.1.6

Systems of Equations — Further Examples

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

What it means for you:

You'll get more practice solving systems of equations and interpreting the solutions.

Key words:

- dependent
- independent
- inconsistent
- system of linear equations

Check it out:

“Interpreting the solution” means stating whether the system is independent (1 solution), inconsistent (0 solutions), or dependent (infinitely many solutions).

Check it out:

The lines on the graph have identical slopes and different intercepts, so they're parallel.

There's nothing new to learn in this Topic — just practice at using the methods you've learned to solve systems of equations.

More Practice at Solving Systems of Equations

Example 1

Find and interpret the solution of this system of equations: $y + x = 2$
 $y + x = -2$

Solution

Step 1: Graph both equations in the same coordinate plane.

Line of first equation:

$$y + x = 2$$
$$y = -x + 2$$

x	y
0	2
4	-2

The line goes through (0, 2) and (4, -2).

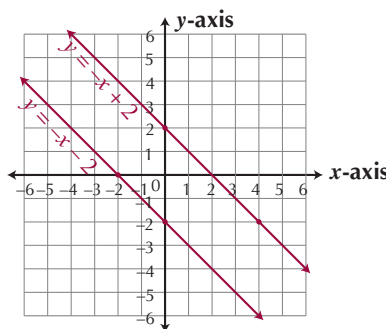
Line of second equation:

$$y + x = -2$$
$$y = -x - 2$$

x	y
0	-2
-2	0

The line goes through (0, -2) and (-2, 0).

Draw the graph:



Step 2: Read off the coordinates of the point of intersection.

The lines are parallel, so there's no point of intersection.

The system of equations has no solutions — the system is inconsistent.

Example 2

Solve this system of equations and state whether the system is independent, inconsistent, or dependent:

$$y - 3x = -5 \quad (\text{Equation 1})$$
$$3x - 4y = -7 \quad (\text{Equation 2})$$
Solution

Step 1: Solve for y in Equation 1.

$$y - 3x = -5$$
$$y = 3x - 5 \quad (\text{Equation 3})$$

Step 2: Substitute $3x - 5$ for y in Equation 2. Then solve to find the value of x .

$$3x - 4y = -7$$
$$3x - 4(3x - 5) = -7$$
$$3x - 12x + 20 = -7$$
$$-9x = -27$$
$$x = 3$$

Step 3: Substitute 3 for x in Equation 3 to find y .

$$y = 3x - 5$$
$$y = 3(3) - 5$$
$$y = 9 - 5$$
$$y = 4$$

So $x = 3, y = 4$ is the solution of the system of equations.

Check by substituting the solution in the original equations.

$$y - 3x = -5$$
$$4 - 3(3) = -5$$
$$4 - 9 = -5$$
$$-5 = -5 \text{ — True statement}$$

$$3x - 4y = -7$$
$$3(3) - 4(4) = -7$$
$$9 - 16 = -7$$
$$-7 = -7 \text{ — True statement}$$

The solution is correct.

The system is independent since it has one solution.

✓ Guided Practice

Solve each system of equations in Exercises 1–6 by either substitution or graphing. If a system isn't independent then say whether it is inconsistent or dependent.

1. $2a - 7b = -12$ and $-a + 4b = 7$

2. $3x - y = -2$ and $x - y = 0$

3. $y + x = 6$ and $3y - x = 6$

4. $y + 2x = -1$ and $3y + 4x = 3$

5. $5y - 4x = 20$ and $y - \frac{4}{5}x = 4$

6. $5y + 3x = -5$ and $y - x = -1$

✓ Independent Practice

Solve each system of equations in Exercises 1–20 by either substitution or graphing. If a system isn't independent then say whether it is inconsistent or dependent.

1. $y = 3x + 6$ and $\frac{1}{4}y + \frac{1}{4}x = \frac{1}{2}$

2. $2x - y = -2$ and $-2x + y = 2$

3. $2x + y = 5$ and $2x + y = 1$

4. $x = 2y + 5$ and $-2x + 4y = 2$

5. $2x - 4y = 4$ and $2y - 2x = -5$

6. $7x - y = 2$ and $2y - 14x = -4$

7. $-3x + 9y = -39$ and $2x - y = 21$

8. $4x + 5y = 2$ and $y = -\frac{4}{5}x - 10$

9. $5x - y = -13$ and $-3x + 8y = 67$

10. $-5x + 9y = -4$ and $-10x + 18y = -1$

11. $6x + 6y = -6$ and $11y - x = 25$

12. $-4x + 7y = 15$ and $8x - 14y = 37$

13. $-7x + 5y = 20$ and $14x - 10y = -40$

14. $5x + 6y = 5$ and $-5x + 3y = 1$

15. $-4x + 7y = 9$ and $5x - 2y = -18$

16. $5x - y = -15$ and $-3x + 9y = 93$

17. $-4x + 9y = 57$ and $5x + 5y = 75$

18. $-2x + 5y = 33$ and $10x - 25y = 69$

19. $-3x + 4y = 41$ and $-12x + 16y = -1$

20. $-2x - 5y = 75$ and $4x + 10y = -150$

Round Up

That was a long Section, with lots of different methods. In Section 5.2 you'll learn one more way of solving systems of equations algebraically, and then in Section 5.3 you'll use all these methods to solve some real-life problems.

Section 5.2

Systems of Equations — Elimination

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

What it means for you:

You'll learn about the elimination method and then use it to solve systems of linear equations in two variables.

Key words:

- system of linear equations
- elimination method

Check it out:

The size (absolute value) is the important thing here. It doesn't matter if the coefficients have opposite signs — for example, -5 and 5 .

Here's one final method of solving systems of linear equations.

Solving Using the Elimination Method

The graphing method used in Section 5.1 **isn't generally very accurate** when the numbers involved are **large** or are **fractions** or **decimals**. It's pretty difficult to get **exact fractions** or decimals from a graph.

In these cases, it's simpler to use the **elimination method**.

The Elimination Method

- Multiply or divide one or both equations so that one variable has coefficients of the same size in both equations.
- Get rid of (eliminate) this variable by either adding the equations or subtracting one equation from the other.

Example 1

Solve this system of equations:

$$\begin{aligned} -3y + 4x &= 11 \\ 5y - 2x &= 5 \end{aligned}$$

Solution

Step 1: Make the coefficients of one variable have the **same absolute value** in each equation.

In this case it is easier to make the x -coefficients the same size, rather than the y -coefficients. Multiplying the second equation by 2 will give you a $-4x$ to match the $4x$ in the first equation.

$$\begin{aligned} 2(5y - 2x) &= 2(5) \\ 10y - 4x &= 10 \end{aligned}$$

Now you have two equations whose x -coefficients have the same absolute value:

$$\begin{aligned} -3y + 4x &= 11 \\ 10y - 4x &= 10 \end{aligned}$$

Step 2: Add the two equations OR subtract one from the other to **eliminate** a variable.

In this case, the coefficients of x are opposites of each other, so adding the equations will eliminate x .

$$\begin{array}{r} -3y + 4x = 11 \\ + 10y - 4x = 10 \\ \hline 7y = 21 \Rightarrow y = 3 \end{array}$$

Example 1 continued

Step 3: Now choose either of the original equations and **substitute** in the value you have found.

Substitute 3 for y , and solve for x .

$$-3y + 4x = 11$$

$$-3(3) + 4x = 11$$

$$-9 + 4x = 11$$

$$4x = 20$$

$$x = 5$$

So $x = 5, y = 3$ is the solution to the system of equations.

It's always a good idea to check your solution in the other equation too.

$$5y - 2x = 5$$

$$5(3) - 2(5) = 5$$

$$15 - 10 = 5$$

$$5 = 5 \text{ — True statement}$$

✓ Guided Practice

Solve the following systems of equations by elimination.

1. $5y - 3x = 19$

$$4y + 3x = -1$$

3. $3y - 2x = 13$

$$2y + x = 4$$

5. $3y + 2x = 23$

$$4y - x = 16$$

7. $4y + 3x = 1$

$$5y - 2x = -16$$

9. $7y - 3x = -11$

$$2y + x = -5$$

2. $5y + 3x = -5$

$$y - x = -1$$

4. $3y - 2x = -4$

$$2y + 3x = -7$$

6. $7y - 2x = -29$

$$2y + 5x = 14$$

8. $6y - 5x = 9$

$$5y + 3x = 29$$

10. $8y + 5x = -14$

$$5y - 3x = -21$$

Practice Using the Elimination Method

In this example, you've got to **subtract** the equations to **eliminate a variable**, rather than add them.

Example 2

Solve this system of equations: $5a + 3b = 19$

$$3a + 2b = 12$$

Solution

Step 1: Make the coefficients of one variable the same in both equations.

To make the coefficients of variable b the same, multiply the first equation by 2 and the second equation by 3.

$$2(5a + 3b = 19) \Rightarrow 10a + 6b = 38$$

$$3(3a + 2b = 12) \Rightarrow 9a + 6b = 36$$

Now you have two equations that have the same coefficient of b .

Example 2 continued

Step 2: Add or subtract the two equations to eliminate a variable.

In this case, the coefficients of b are both positive, so subtracting one equation from the other will eliminate b .

$$\begin{array}{r} 10a + 6b = 38 \\ - 9a + 6b = 36 \\ \hline a = 2 \end{array}$$

Step 3: Now choose either of the original equations and substitute in the value you found.

Substitute 2 for a , and solve for b .

$$\begin{array}{r} 3a + 2b = 12 \\ 3(2) + 2b = 12 \\ 6 + 2b = 12 \\ 2b = 6 \\ b = 3 \end{array}$$

Therefore $a = 2$, $b = 3$ is the solution to the system of equations.

Check your solution in the other equation:

$$\begin{array}{r} 5a + 3b = 19 \\ 5(2) + 3(3) = 19 \\ 10 + 9 = 19 \\ 19 = 19 \text{ — True statement} \end{array}$$

Check it out:

It's always a good idea to write both parts of the solution together clearly.

You have a few choices to make when you're using the elimination method — such as which variable to **eliminate**, which equation to **subtract** from the other, and which of the original equations to **substitute** your result into.

Always try to make things easy for yourself by keeping the algebra as **simple** as possible.

Example 3

Solve:
$$\begin{array}{r} 3y + 6x = -6 \\ 4y - 2x = 7 \end{array}$$

Solution

Step 1: Make the x -coefficients the same size by multiplying the second equation by 3.

$$\begin{array}{r} 3(4y - 2x = 7) \\ 12y - 6x = 21 \end{array}$$

Now you have two equations with x -coefficients of 6 and -6 :

$$3y + 6x = -6 \text{ and } 12y - 6x = 21.$$

Example 3 continued

Step 2: Add these equations together to eliminate x .

$$\begin{array}{r} 3y + 6x = -6 \\ + 12y - 6x = 21 \\ \hline 15y = 15 \Rightarrow y = 1 \end{array}$$

Step 2: Substitute 1 for y in one of the original equations and solve for x .

$$\begin{array}{r} 4y - 2x = 7 \\ 4(1) - 2x = 7 \\ 4 - 2x = 7 \\ -2x = 3 \\ x = -1.5 \end{array}$$

So $x = -1.5, y = 1$ is the solution to the system of equations.

Check the solution in the other equation:

$$\begin{array}{r} 3y + 6x = -6 \\ 3(1) + 6(-1.5) = -6 \\ 3 - 9 = -6 \text{ — True} \end{array}$$

Example 4

Solve this system of equations: $\frac{5}{8}a - \frac{1}{2}b = \frac{1}{4}$ and $\frac{2}{3}a - \frac{3}{5}b = \frac{2}{15}$

Solution

This example has **fractional coefficients**. To make the equations easier to solve, first multiply each equation by the LCM of the denominators to convert the fractional coefficients to **integers**.

$$8\left(\frac{5}{8}a - \frac{1}{2}b = \frac{1}{4}\right) \Rightarrow 5a - 4b = 2$$

$$15\left(\frac{2}{3}a - \frac{3}{5}b = \frac{2}{15}\right) \Rightarrow 10a - 9b = 2$$

Now you can solve these equations using the same method as the previous examples:

Step 1: Make the a -coefficients the same by multiplying the first equation by 2.

$$\begin{array}{r} 2(5a - 4b = 2) \\ 10a - 8b = 4 \end{array}$$

Now you've got two equations that have the same a -coefficient: $10a - 8b = 4$ and $10a - 9b = 2$.

Step 2: Subtract one equation from the other to eliminate a .

$$\begin{array}{r} 10a - 8b = 4 \\ - 10a - 9b = 2 \\ \hline b = 2 \end{array}$$

Check it out:

LCM of 8, 2, and 4 is 8.
LCM of 3, 5, and 15 is 15.

Check it out:

Subtracting the second equation from the first means that there are no negative terms in the resulting equation.

Check it out:

You could also substitute into one of the equations in which the fractional coefficients have been converted to integers.

Example 4 continued

Step 3: Substitute 2 for b in one of the original equations and solve for a .

$$\frac{5}{8}a - \frac{1}{2}b = \frac{1}{4} \Rightarrow \frac{5}{8}a - 1 = \frac{1}{4} \Rightarrow \frac{5}{8}a = \frac{5}{4} \Rightarrow 5a = 10 \Rightarrow a = 2$$

So $a = 2$, $b = 2$ is the solution to the system of equations.

Check the solution in the other equation:

$$\frac{2}{3}a - \frac{3}{5}b = \frac{2}{15} \Rightarrow \frac{4}{3} - \frac{6}{5} = \frac{2}{15} \Rightarrow \frac{20}{15} - \frac{18}{15} = \frac{2}{15} \quad \text{— True}$$

✓ Guided Practice

Solve the following systems of equations by elimination.

11. $2a - 5b = 3$

$$4a - 4b = -12$$

13. $2y = -x - 11$

$$2y - x = -5$$

15. $7a + 2b = 10$

$$3a + b = 1$$

17. $\frac{1}{2}y - \frac{1}{3}x = -\frac{7}{4}$

$$\frac{2}{5}y + \frac{2}{3}x = \frac{7}{15}$$

12. $4y - 5x = 7$

$$2y - 3x = -1$$

14. $7y - 2x = 3$

$$6y - 5x = -4$$

16. $5a - 2b = 3$

$$3a - 5b = 17$$

18. $\frac{4}{3}a + \frac{3}{2}b = 4$

$$\frac{1}{2}a + \frac{1}{4}b = -1$$

✓ Independent Practice

Solve the following systems of equations by elimination.

1. $10y - 2x = 3$

$$6y + 4x = 7$$

3. $\frac{5}{6}a + \frac{2}{3}b = -3\frac{1}{2}$

$$2a + \frac{1}{2}b = -4$$

5. $0.4m - 0.9c = -0.1$

$$0.3m + 0.2c = 0.8$$

7. $5y - 4x = -6.1$

$$8y + 5x = -6.625$$

9. $0.7y - 0.4x = -4.7$

$$0.9y + 0.5x = -3$$

2. $3m - 4c = 1$

$$6m - 6c = 5$$

4. $\frac{1}{2}a - \frac{1}{3}b = -1$

$$\frac{1}{3}a + \frac{3}{7}b = \frac{9}{7}$$

6. $y - 0.4x = -1.8$

$$y + 0.2x = -0.6$$

8. $0.7y - 0.3x = 2.7$

$$0.4y + 0.3x = 0.6$$

10. $0.3c + 0.1d = 0.2$

$$0.4c + 0.6d = -0.2$$

11. Find x and y , by first finding a and c using the following systems of equations:

$$0.3a - 0.7b = -0.5 \quad \text{and} \quad 1.7a + 3b = 11.1$$

$$1.2c + 0.3d = 3.9 \quad \text{and} \quad -0.2c - 0.4d = 0.4$$

$$ax + cy = 6 \quad \text{and} \quad 3ax - 2cy = -42$$

Round Up

The elimination method is usually *more reliable* than trying to solve complicated systems of equations graphically. Just be really careful with the *signs* when you're subtracting.

Topic 5.3.1

Section 5.3 Systems of Equations — Applications

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

15.0: Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

What it means for you:

You'll solve real-life problems involving systems of linear equations.

Key words:

- system of linear equations
- substitution method
- elimination method

Check it out:

$8.50c$ is the amount spent on CDs.

$12.50d$ is the amount spent on DVDs.

Now it's time to use *all the systems of equations methods from Sections 5.1 and 5.2 to solve some real-life problems.*

Systems of Equations Help Solve Real-Life Problems

You can use **systems of linear equations** to represent loads of **real-life problems**.

Once you've written a system of equations, you can then use either **substitution** or **elimination** to solve the system of equations and therefore **solve the problem**.

You Need the Same Two Variables in Each Equation

Example 1

A store has a year-end sale on CDs and DVDs. Each CD is reduced to \$8.50, and the DVDs are going for \$12.50 each. Akemi bought a total of 15 items (some CDs and some DVDs) for \$163.50. Determine how many CDs and DVDs she bought.

Solution

One way to find the number of CDs and DVDs is to set up a system of two linear equations in two variables.

Represent the unknown values with variables —

Let c = number of CDs Akemi bought

d = number of DVDs Akemi bought

Since Akemi bought a total of 15 CDs and DVDs:

$$c + d = 15 \text{ — this is your first equation}$$

Now, consider the fact that each CD cost \$8.50, each DVD cost \$12.50, and Akemi spent \$163.50. This leads to the equation:

$$8.50c + 12.50d = 163.50 \text{ — this is your second equation}$$

You now have a system of two linear equations in two variables, c and d .

Check it out

This part is nothing new — it is the elimination method you used in Section 5.2.

Don't forget

Always remember to answer the original question clearly, in words.

Example 1 continued

$$\begin{aligned} \text{Now solve your system of equations:} \quad c + d &= 15 \\ 8.50c + 12.50d &= 163.50 \end{aligned}$$

Step 1: Make the coefficient of c the same in both equations. Do this by multiplying the first equation ($c + d = 15$) by 8.50.

$$\begin{aligned} 8.50(c + d) &= 15 \cdot 8.50 \\ 8.50c + 8.50d &= 127.50 \end{aligned}$$

Now you have two equations that have the same coefficient of c :
 $8.50c + 12.50d = 163.50$ and $8.50c + 8.50d = 127.50$

Step 2: Subtract one equation from the other to eliminate c .

$$\begin{array}{r} 8.50c + 12.50d = 163.50 \\ - 8.50c + 8.50d = 127.50 \\ \hline 4d = 36 \end{array} \Rightarrow d = 9$$

Step 3: Substitute $d = 9$ into one of the original equations.

$$\begin{aligned} c + d &= 15 \\ c + 9 &= 15 \\ c &= 6 \end{aligned}$$

So Akemi bought 6 CDs and 9 DVDs.

Check the solution in the other equation:

$$\begin{aligned} 8.50c + 12.50d &= 163.50 \\ 8.50(6) + 12.50(9) &= 163.50 \\ 163.50 &= 163.50 \quad \text{— True} \end{aligned}$$

Guided Practice

1. Pedro bought a total of 18 paperback and hardcover books for a total of \$150. If each paperback was on sale for \$6.50 and the hardcovers were on sale for \$9.50 each, calculate how many paperbacks and how many hardcovers Pedro bought.
2. Three cans of tuna fish and four cans of corned beef cost \$12.50. However, six cans of tuna fish and three cans of corned beef cost \$15. How much does each type of can cost individually?
3. A school raised funds for its sports teams by selling tickets for a play. A ticket cost \$5.75 for adults and \$2.25 for students. If there were five times as many adults at the play than students and ticket sales raised \$2480, how many adult and student tickets were sold?
4. Teresa bought five cups and four plates for \$14.50. However, five plates and four cups would have cost \$14.75. Find the cost of each item individually.
5. Simon has a total of 21 dimes and quarters in his coin bank. If the value of the coins is \$4.05, how many coins of each type does Simon have?

Using Systems of Equations to Solve Age Questions

Example 2

The sum of Jose's and Elizabeth's ages is 40 years. Five years ago, Elizabeth was four times as old as Jose. How old are Jose and Elizabeth now?

Solution

First form a system of two equations.

Present Ages:

Let x = Jose's age

y = Elizabeth's age

The sum of their ages is 40 $\Rightarrow x + y = 40$

Ages 5 years ago:

$x - 5$ = Jose's age

$y - 5$ = Elizabeth's age

Elizabeth's age was four times Jose's age

$\Rightarrow y - 5 = 4(x - 5) \Rightarrow y - 5 = 4x - 20 \Rightarrow y - 4x = -15$

So the system of equations is: $x + y = 40$

$$y - 4x = -15$$

Now solve the system of equations. The y -coefficients are the same, so subtract one equation from the other to eliminate y :

$$\begin{array}{r} y + x = 40 \\ - y - 4x = -15 \\ \hline 5x = 55 \Rightarrow x = 11 \end{array}$$

Now substitute 11 for x in an original equation.

$$x + y = 40$$

$$11 + y = 40$$

$$y = 29$$

Therefore Jose is 11 years old and Elizabeth is 29 years old.

Check the solution in the other equation.

$$y - 4x = -15$$

$$29 - 4(11) = -15$$

$$-15 = -15 \text{ — True}$$

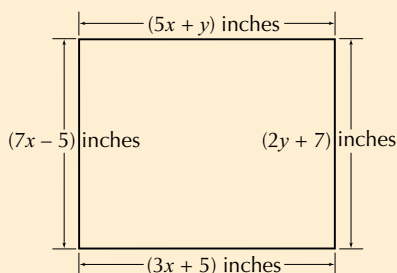
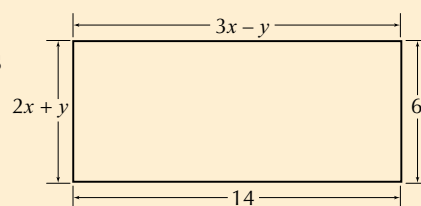
✓ Guided Practice

- The sum of Julio's and Charles's ages is 44 years. Seven years from now, Julio will be twice Charles's age now. How old is each one now?
- A mother is five times as old as her daughter. In five years, the mother will be three times as old as her daughter. How old is each now?
- Anthony is twice as old as Teresa. In 20 years, the sum of their ages will be 43. How old is each now?
- The sum of Ivy's and Audrey's ages is 27. Nine years ago, Ivy was twice as old as Audrey. How old is each now?
- Three years from now, a father will be 10 times as old as his daughter. One year ago, the sum of their ages was 36. How old is each now?

✓ Independent Practice

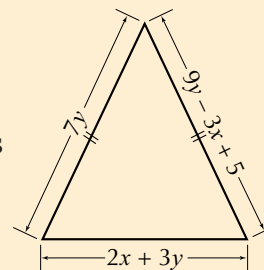
- Eight goldfish and 12 angelfish cost \$53. Five goldfish and eight angelfish cost \$34.75. If all goldfish are the same price and all angelfish are the same price, find the price of each goldfish and each angelfish.
- Seven pairs of shorts and nine shirts cost \$227.95. Three pairs of shorts and five shirts cost \$116.55. If all pairs of shorts are the same price and all shirts are the same price, how much does each pair of shorts and each shirt cost?

- Find the values of x and y if the figure shown on the right is a rectangle.



- By finding the values of x and y , calculate the area of the rectangle on the left. All dimensions are in inches.

- Given that the triangle on the right is an isosceles triangle with a 40 cm perimeter, find the values of x and y .



Round Up

Systems of linear equations are really useful for working out *real-life problems*.

In the rest of this Section you'll see more everyday situations *modeled as systems of equations*.

Systems of Equations — Integer Problems

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

15.0: Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

What it means for you:

You'll solve integer problems involving systems of linear equations.

Key words:

- system of linear equations
- substitution method

In this Topic you'll solve systems of linear equations to figure out solutions to problems involving integers.

Using Systems of Equations to Solve Integer Problems

Example 1

The sum of two integers is 53. The larger number is 7 less than three times the smaller one. Find the numbers.

Solution

First form a system of two equations.

Let x = smaller number

y = larger number

The sum of the two integers is 53, so $x + y = 53$.

The larger is 7 less than 3 times the smaller one, so $y = 3x - 7$.

So the system of equations is $x + y = 53$ and $y = 3x - 7$.

Now solve the system of equations. The variable y in the second equation is already expressed in terms of x , so it makes sense to use the substitution method.

Substitute $3x - 7$ for y in the first equation.

$$\begin{aligned}x + y &= 53 \\x + (3x - 7) &= 53 \\4x - 7 &= 53 \\4x &= 60 \\x &= 15\end{aligned}$$

Now, substitute 15 for x in the equation $y = 3x - 7$.

$$\begin{aligned}y &= 3x - 7 \\y &= 3(15) - 7 \\y &= 45 - 7 \\y &= 38\end{aligned}$$

That means that the integers are 15 and 38.

The solution should work in both equations:

$$\begin{aligned}x + y = 53 &\Rightarrow 15 + 38 = 53 \Rightarrow 53 = 53 \text{ — True} \\y = 3x - 7 &\Rightarrow 38 = 3(15) - 7 \Rightarrow 38 = 38 \text{ — True}\end{aligned}$$

A useful check is to make sure that the answer matches the information given in the question:

- The sum of the two integers is $15 + 38 = 53$. This matches the question.
- Three times the smaller integer is $15 \times 3 = 45$. The larger integer is $45 - 38 = 7$ less than this.

Guided Practice

1. The sum of two numbers is 35 and the difference between the numbers is 13. Find the numbers.
2. The sum of two integers is 6 and the difference between the numbers is 40. Find the numbers.
3. The difference between two numbers is 50. Twice the higher number is equal to three times the lower number. Find the numbers.
4. The difference between two numbers is 11. If the sum of twice the higher number and three times the lower number is 137, find the numbers.
5. There are two numbers whose sum is 64. The larger number subtracted from four times the smaller number gives 31. Find the two numbers.

Independent Practice

1. The length of a rectangle is three times the width. The perimeter is 24 inches. Find the length and the width.
2. In a soccer match, Team A defeated Team B by 3 goals. There were a total of 13 goals scored in the game. How many goals did each team score?
3. An outdoor adventure club has 35 members, who are all either rock climbers or skiers. There are 3 more rock climbers than there are skiers. How many climbers are there?
4. The tens digit of a two-digit number is three times the ones digit. If the sum of the digits is 12, find the two-digit number.
5. The sum of the digits of a two-digit number is 13. The number is 27 more than the number formed by interchanging the tens digit with the ones digit. Find the number.
6. Subtracting the ones digit from the tens digit of a positive two-digit number gives -2 . Find the number given that the number is 18 less than the number formed by reversing its digits. (More than one answer is possible.)
7. The ones digit of a positive two-digit number is seven more than the tens digit. If the number formed by reversing the digits is 63 more than the original number, find the original number. (More than one answer is possible.)
8. In a two-digit number, the sum of the digits is 10. Find the number if it is 36 more than the number formed by reversing its digits.

Round Up

For questions like this, it doesn't matter which letters you choose to *represent the unknown quantities*. It's a good idea to pick letters that *remind you* in some way of what they *represent*. For example, in Example 1 involving CDs and DVDs, the letters c and d were sensible choices.

Systems of Equations — Percent Mix Problems

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

15.0: Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

What it means for you:

You'll solve percent mix problems involving systems of linear equations.

Key words:

- percent mix
- system of linear equations

Don't forget:

5% is $\frac{5}{100}$, so the number of gallons of real fruit juice in a gallons of apple drink is $0.05a$.

Percent mix problems are questions that involve two different amounts being mixed together to make a single mixture.

Percent Mix Problems Can Be Systems of Equations

Example 1

A high school sports coach decided to give a year-end party for all students at the school. The cafeteria manager decided to make 50 gallons of fruit juice drink by mixing some apple drink that contains 5% real fruit juice with strawberry drink that contains 25% real fruit juice.

If the 50-gallon fruit juice drink is 10% real fruit juice, how many gallons of apple drink and strawberry drink did the cafeteria manager use?

Solution

First form a system of two equations.

Let a = gallons of apple drink used

s = gallons of strawberry drink used

The total number of gallons is 50, so: $a + s = 50$

Now consider the total real fruit juice in the drink:

$$(5\% \text{ of } a) + (25\% \text{ of } s) = 10\% \text{ of } 50$$

$$0.05a + 0.25s = 0.10(50)$$

$$0.05a + 0.25s = 5$$

$$5a + 25s = 500$$

$$a + 5s = 100$$

So the system of equations is: $a + s = 50$

$$a + 5s = 100$$

Now solve the system of equations.

The coefficient of a is the same in both equations, so subtract one equation from the other to eliminate a .

$$\begin{array}{r} a + 5s = 100 \\ - a + s = 50 \\ \hline 4s = 50 \Rightarrow s = 12.5 \end{array}$$

Substitute 12.5 for s in one of the original equations.

$$\begin{array}{r} a + s = 50 \\ a + 12.5 = 50 \\ \hline a = 37.5 \end{array}$$

That means that **37.5 gallons of apple drink was used and 12.5 gallons of strawberry drink.**

Example 1 continued

Check the solution in the other equation.

$$\begin{aligned}a + 5s &= 100 \\37.5 + 5(12.5) &= 100 \\100 &= 100 \text{ — True}\end{aligned}$$

The answer must also match the information given in the question:

- 5% of 37.5 gallons of apple drink is real fruit juice.
This is $0.05 \times 37.5 = 1.875$ gallons.
- 25% of 12.5 gallons of strawberry drink is real fruit juice.
This is $0.25 \times 12.5 = 3.125$ gallons.

The total fruit juice is $1.875 + 3.125 = 5$ gallons.

5 gallons is 10% of the 50-gallon mixture, which matches the question.

Guided Practice

1. A football coach calls a total of 50 plays in a game, some running plays and some passing plays. If 60% of the plays she calls are running plays, how many passing plays did she call?
2. During a fund-raiser, a girl collected a total of 60 quarters and half-dollars. If she raised \$21.25, how many coins of each type did she collect?
3. A mother wants to make a quart of 2% fat milk for her children. She has a gallon of 1% fat milk and a gallon of whole milk (4% fat). How much of each should she mix to get a quart of 2% fat milk?
4. A ranger had 150 gallons of 60%-pure disinfectant that he mixed with 80%-pure disinfectant until the mixture was 70%-pure disinfectant. How much 80%-pure disinfectant was used by the ranger?
5. Two brands of tea are worth 60 cents per pound and 90 cents per pound respectively. How many pounds of each tea must be mixed to produce 138 pounds of a mixture that would be worth 80 cents per pound?
6. A pharmacist has a bottle of 10% boric acid and a bottle of 6% boric acid. A prescription requires 50 milliliters of a 7% boric acid solution. What volume of each solution should the pharmacist mix to get the desired solution?
7. In an industrial process, milk containing 4% butterfat is mixed with cream that contains 44% butterfat to obtain 250 gallons of 10% butterfat milk. How many gallons of 4% butterfat milk and 44% butterfat cream must be mixed to obtain the 250 gallons of 10% butterfat milk?

Independent Practice

1. A company invested a total of \$5000, some at 6% and the rest at 8% per year. If the total return from the investments after one year was \$330, how much money was invested at each rate?
2. Stacey used her 20% discount card to buy 7 cans of tomatoes and 15 cans of mushrooms for \$10.64. The next day Brian spent \$7.21 on 17 cans of tomatoes and 5 cans of mushrooms in a “30% off” sale. Find the undiscounted pre-sale price of each type of can.
3. Teresa’s piggy bank has a total of 100 dimes and nickels. If the piggy bank has a total of \$7.50, how many of each coin does she have?
4. Anthony invested \$1000 in two stocks. Stock A increased in value by 20%, while Stock B decreased in value by 10%. If Anthony ended the year with \$1160 worth of stocks, how much money did he initially invest in each?
5. Dr. Baines makes a 5% metal hydride by mixing Material A, which contains 3% hydride, with Material K, which contains 7% hydride. If the final mixture weighs 10 kg, how much of each material does he mix?
6. Apples cost \$0.75 per pound and pears cost \$0.40 per pound. How much of each would need to be mixed to make 5 pounds of fruit salad worth \$0.50 per pound?
7. Maddie invested \$5000, some at a 5% rate of return and the rest at a 2% rate of return per year. If the total return from the investments after one year was \$200, how much money was invested at each rate?
8. Stephen collected a total of 75 quarters and half-dollars. If he collected \$30.50, how many coins of each type did he collect?
9. Robert used his 10% discount card to buy 4 pizzas and 2 calzones for \$45. The next day Audrey spent \$15 on 1 pizza and 2 calzones in a “25% off” sale. Find the undiscounted price of pizzas and calzones.
10. Fred invested \$6000, some at a 10% annual return and the rest at a 4% annual return. If the total return from the investments after one year was \$400, how much money was invested at each rate?

Round Up

Percent mix problems *don't* have to involve *liquids* like the example in this Topic. You could mix together *solids* or even amounts of *money* — the solving method is always the *same*.

Systems of Equations — Rate Problems

California Standards:

9.0: Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

15.0: Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.

What it means for you:

You'll solve rate problems involving systems of linear equations.

Key words:

- rate
- system of linear equations

Check it out:

When the boat travels downriver, it's helped along by the water current. When the boat travels upstream, it's slowed down by the water current.

Here's the final application of systems of linear equations — rate problems deal with speed, distance, and time.

Rate Problems Can Be Systems of Equations Too

Example 1

During a storm, a flood rescue team in a boat takes 3 hours to travel downriver along a 120-mile section of Nastie river.

If it takes 4 hours to travel the same river section upriver, find the boat's speed in still water and the speed of the water current. [Assume the boat has the same speed relative to the water and that the speed of the current remains constant.]

Solution

First form a system of two equations.

$$\begin{aligned} \text{Let } x &= \text{boat's speed in still water} \\ y &= \text{water current speed} \end{aligned}$$

Downriver:

When traveling downriver, the water speed adds to the boat speed.

So **boat speed downriver** = $x + y$

Use the formula **Distance = Speed × Time**:

$$120 \text{ miles} = (x + y) \times 3 \text{ hours}$$

$$120 = 3(x + y)$$

$$\frac{120}{3} = x + y$$

$$40 = x + y$$

Upriver:

When traveling upriver, the water speed acts against the boat speed.

So **boat speed upriver** = $x - y$

Use **Distance = Speed × Time**:

$$120 \text{ miles} = (x - y) \times 4 \text{ hours}$$

$$120 = 4(x - y)$$

$$\frac{120}{4} = x - y$$

$$30 = x - y$$

So the system of equations is: $40 = x + y$
 $30 = x - y$

Example 1 continued

Now solve the system of equations.

The y -coefficients have opposite values, so add the equations to eliminate y .

$$\begin{array}{r} 40 = x + y \\ + 30 = x - y \\ \hline 70 = 2x \end{array} \Rightarrow x = 35$$

Now substitute 35 for x in an original equation.

$$\begin{array}{r} 40 = x + y \\ 40 = 35 + y \\ y = 5 \end{array}$$

So, the boat's **speed in still water (x) is 35 mph** and the **water current speed (y) is 5 mph**.

Check the solution in the other equation.

$$\begin{array}{r} 30 = x - y \\ 30 = 35 - 5 \\ 30 = 30 \text{ — True} \end{array}$$

Check it out:

The distance was measured in miles and the time was measured in hours, so the units must be miles per hour (mph).

Guided Practice

1. The Lee High School crew team takes 2 hours to row downriver along a 60-mile section of the Potomac River. It takes 4 hours to travel the same river section upriver. Find the boat's speed in still water and the speed of the water current.
2. A plane travels 300 miles in 40 minutes against the wind. Flying with the wind, the same plane travels 200 miles in 20 minutes. Find the plane's speed without wind and the wind speed.
3. Sarah rides her bicycle to work, a journey of five miles (one way). One day the journey to work is against the wind, and takes 30 minutes. The ride back, with the wind, takes 20 minutes. Find the wind speed, and work out how long the ride (one way) would take with no wind.
4. A red car and a blue car start at the same time from towns that are 16 miles apart, and travel towards each other. The red car is 7 mph faster than the blue car. After 15 minutes the cars are 7 miles apart. Find the speed of each car.

Independent Practice

1. It takes a canoeist 1.5 hours to paddle down a 10-mile stretch of river. If it takes 3 hours to travel the same river section upriver, find the canoeist's speed in still water and the speed of the water current. Round answers to the nearest tenth.
2. A medical helicopter pilot flies 200 miles with a tailwind in 2 hours. On the return trip, it takes 2.5 hours to fly against the wind. Find the speed of the helicopter in no wind and the speed of the wind.
3. A small powerboat travels down a 24-mile stretch of river in one hour. If it takes 3 hours to travel the same river section upriver, find the powerboat's speed in still water and the speed of the water current.
4. A plane flies 700 miles with a tailwind in 5 hours. On the return trip, it takes 6 hours to fly the same 700 miles against the wind. Find the speed of the plane in no wind and the speed of the wind. Round answers to the nearest tenth.
5. Two planes start at the same time from cities that are 2500 miles apart, and travel toward each other. The rate of one plane exceeds the rate of the other by 15 mph. After 4 hours the airplanes were 1000 miles apart. Find the rate of each airplane.
6. Kyle lives in Washington, DC and Robert lives in Boston. The cities are 440 miles apart. They start driving toward each other at the same time. Kyle is driving an average of 5 mph faster than Robert. After 5 hours the cars are 180 miles apart. Find the rate of each car.
7. Sheri and Peter start running towards each other at the same time, from their houses that are 4000 feet apart. Sheri is running 2 feet per minute faster than Peter. After 5 minutes, they are still 300 feet apart. Find the running speeds of Sheri and Peter.
8. Casey leaves home at 10 a.m. and drives at an average speed of 25 mph. Marshall leaves the same house 15 minutes later, and drives the same route, but twice as fast as Casey. At what time will Marshall pass Casey, and how far will they be from home when he does?
9. Pittsburgh is 470 miles from Chicago and 350 miles from Philadelphia. Trains leave Chicago and Philadelphia at the same time, but the Chicago train travels 40 mph faster than the Philadelphia one. Both trains reach Pittsburgh at the same time. Find each train's speed, rounding answers to the nearest tenth.
10. At 07:00 a train leaves a station, traveling at 55 mph. At 07:30 an express train leaves the same station, traveling the same route at 70 mph. How long will it take the express train to overtake the other train, and how far will they be from the station when it does?

Round Up

*The word problems in this Section all look quite different — but you use the **same methods** each time to solve them. If you're having trouble with solving them, look back at Sections 5.1 and 5.2 and try going through some examples that **don't** involve real-life situations.*

Chapter 5 Investigation

Breaking Even in an Egg Business

Even though this Investigation looks tough because it's about money, it's just *systems of equations*.



Part 1:

You decide to set up an **egg-selling business**.

You plan to keep **12 hens** in your yard and sell the eggs from your home.

The **initial costs** of setting up the business are \$47 for a hen house and \$3 for each hen.

The hens will cost a total of **50 cents a day** to feed and you think each hen will lay **one egg per day**. You plan to sell the eggs for **20 cents each**.

1) How many days will it be before you **break even**?

Assume you manage to sell all the eggs laid without any going to waste.

2) On one set of axes, draw **graphs** to show how the **costs** and **amount earned** will change over the first 50 days.

The “break-even” point is when the amount of money you have earned equals the amount of money you’ve spent on the business.

Part 2:

After further research, you discover that this type of hen only lays eggs on **four days out of five**.

However, you can buy special hens for **\$5** each that will not only lay one egg per day, but will lay **two eggs every fourth day**. You have a total of **\$98** to invest in setting up your business.

How many of each type of hen should you buy to maximize your profits?

Remember — the hen house will only accommodate up to **12 hens**.

Extension

1) Your mother is not happy about you using the yard for your business and demands **10%** of your profit. With this in mind, calculate how much money you will make in the **first year**.

2) You are given **\$50** for your birthday. Will you make **more money** over one year if you buy another hen house and one hen or invest it at 6%? Investigate how receiving a **different amount of money** would affect the best option.

Open-ended Extension

Make up a business and describe its **costs** and the amount it **charges** for its products or services. Calculate when you will **break even**.

Exchange business plans with a partner and each devise some **changes** that will affect the other person’s business. Investigate how your partner’s changes will affect your **profits** and **break-even point**.

Your break-even point *doesn't necessarily have to be after a certain number of days* — it might be after you sell a **certain number** of your products. For example, a **milkshake company** might have an initial cost of buying a blender and cups. There will then be the cost of the **ingredients** for each milkshake made. After you sell a certain number of milkshakes you will have earned an amount **equal** to the amount that you’ve spent so far.

Round Up

*If you come across any real-life situation that involves **two or more equations**, you’ll probably have to solve them using **systems of equations** — using all the skills you learned in this Chapter.*

Chapter 6

Manipulating Polynomials

Section 6.1	Adding and Subtracting Polynomials	263
Section 6.2	Multiplying Polynomials	274
Section 6.3	Dividing Polynomials	283
Section 6.4	Special Products of Binomials	297
Section 6.5	Factors	302
Section 6.6	Factoring Quadratics	310
Section 6.7	More on Factoring Polynomials	318
Section 6.8	More on Quadratics	325
Investigation	Pascal's Triangle	331

Topic 6.1.1

Section 6.1 Polynomials

California Standards:

10.0: Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

What it means for you:

You'll learn what polynomials are, and you'll simplify them.

Key words:

- monomial
- polynomial
- like terms
- degree

Check it out:

The word monomial comes from the Greek words: **mono** — meaning one
nomos — meaning part or portion.
So monomial means one term.

Check it out:

The word polynomial comes from the Greek words: **poly** — meaning many
nomos — meaning part or portion.
So polynomial means many terms.

Polynomial — another math word that sounds a lot harder than it actually is. Read on and you'll see that actually polynomials are *not* as complicated as you might think.

A Monomial is a Single Term

A **monomial** is a single-term expression. It can be either a **number** or a **product of a number and one or more variables**.

For example, **13**, **$2x^2$** , and **$-x^3yn^4$** are all monomials.

A Polynomial Can Have More Than One Term

A **polynomial** is an algebraic expression that has one or more terms (each of which is a monomial).

For example, **$x + 1$** and **$-3x^2 + 2x + 1$** are polynomials.

There are a couple of special types of polynomial:

A **binomial** is a **two**-term polynomial, such as $x^2 + 1$.

A **trinomial** is a polynomial with **three** terms, such as $-3x^2 + 2x + 1$.

Guided Practice

For each of the polynomials below, state whether it is a monomial, a binomial, or a trinomial.

- | | |
|------------------------------|---------------------|
| 1. $5x$ | 2. $9x^2 + 4$ |
| 3. $2y$ | 4. $2x^2y$ |
| 5. $2y + 2$ | 6. $5x - 2$ |
| 7. $7x^3$ | 8. $x^2 + y$ |
| 9. $x^2 + 2x + 3$ | 10. $3x^2 + 4x - 8$ |
| 11. $4.3x - 8.9x^2 + 4.2x^3$ | 12. $0.3x^2y$ |
| 13. $0.3x^2y + xy^2 + 4xy$ | 14. 8.7 |
| 15. $a^2 + b^2$ | 16. $97.9a - 14.2c$ |

Use Like Terms to Simplify Polynomials

Like terms are terms that have exactly the **same variables** — for example, $-2x^2$ and $5x^2$ are like terms. Like terms always have the same variables, but may have different coefficients.

A **polynomial** can often be **simplified** by **combining** all **like terms**.

Example 1

Simplify the expression $2x^2 + 4y + 3x^2$.

Solution

Notice that there are two like terms, $2x^2$ and $3x^2$:

You can **combine** the like terms: $2x^2 + 3x^2 = 5x^2$

So $2x^2 + 4y + 3x^2 = 5x^2 + 4y$

Guided Practice

Simplify each of the following polynomials.

17. $x + 1 + 2x$

18. $3y + 2y$

19. $9x^2 + 4x + 7x$

20. $x^2 + x + x^2$

Simplify each of the following polynomials, then state whether your answer is a monomial, binomial, or trinomial.

21. $3x^2 + 4 - 8 + x^2$

22. $8x^3 + x^4 - 6x^3 + 4$

23. $3x^2y - 2x^2y + 8$

24. $7 - 2y + 3 - 10$

25. $4x^3 + 7 - x^3 + 4 - 3x^3 - 11$

26. $5x^2 + 9x^2 + 4 + 2y$

27. $3xy + 4xy + 5x^2y - 4xy^2$

28. $9x^5 + 2x^2 + 4x^4 + 5x^5 - 3x^4 - x^2$

Finding the Degree of a Polynomial

The **degree** of a polynomial in x is the size of the highest power of x in the expression.

If you see the phrase “**a fourth-degree polynomial in x** ,” you know that it will contain at least one **term with x^4** , but it won’t contain any higher powers of x than 4.

For example:

$2x + 1$	has degree 1	— it’s a first-degree polynomial in x
$y^2 + y - 3$	has degree 2	— it’s a second-degree polynomial in y
$x^4 - x^2$	has degree 4	— it’s a fourth-degree polynomial in x

✓ Guided Practice

State the degree of each of the following polynomials.

29. $3x + 5$

30. $3x^4 + 2$

31. $x^2 + 2x + 3$

32. $x + 4 + 2x^2$

Simplify and state the degree of each of the following polynomials.

33. $2x + x^2 + x - 3$

34. $3a^3 + 4a - 2a^3 + 4a^2$

35. $4x^3 + 4x^8 - 3x^8 + 2x^3$

36. $3y + 2y - 5y^2 + 6y$

37. $b^{13} + 2b^{13} - 8 + 4 - 3b^{13}$

38. $z^3 + z^3 - z^6 + z^7 + 3z^7$

39. $c^4 + c^3 + c^3 - c^4 + c - 2c^3$

40. $x - 2x^9 - 8x^4 + 13x^2$

✓ Independent Practice

For the polynomials below state whether they are a monomial, a binomial, or a trinomial.

1. $19a^2 + 16$

2. $2c - 4a + 6$

3. $42xy$

4. $16a^2b + 4ab^2$

Simplify each of the following polynomials.

5. $0.7x^2 + 9.8 - x^2$

6. $17x^2 - 14x^9 + 7x^9 - 7x^2 + 7x^9$

7. $0.8x^4 + 0.3x^2 + 9.6 - x^2 - 9x^4 + 1.6x^2$

State the degree of the following polynomials.

8. $x - 9x^6 + 4$

9. $14x^8 + 16x^{10} + 4x^8$

10. $2x^2 - 4x^4 + 7x^5$

11. $2x^2 - 4x + 8$

Simplify each polynomial, state the degree of the polynomial, and determine whether it is a monomial, a binomial, or a trinomial.

12. $93a^2 + 169 - 4a - 81a^2 + 7$

13. $7.9x^2 - 13x^4 - 1.5x^4 + 1.4x^2$

14. $5x^9 - 6x^9 + 4 + x^9 - 3 - 1$

15. $\frac{1}{2}x^9 - x^3 - x^3 + \frac{1}{3}x^9 + \frac{7}{9}$

16. $\frac{1}{9}x^{10} - \frac{3}{4}x^6 - \frac{3}{5}x^6 + \frac{1}{9}x^{10} - \frac{3}{4}$

17. When a third degree monomial is added to a second degree binomial, what is the result?

18. When a 4th degree monomial is added to a 6th degree binomial, what are the possible results?

Round Up

This Topic gets you started on manipulating polynomials, by simplifying them. In the next couple of Topics you'll see how to add and subtract polynomials.

Adding Polynomials

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

10.0: Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

What it means for you:

You'll add polynomials and multiply a polynomial by a number.

Key words:

- polynomial
- like terms
- inverse

Don't forget:

See Topic 1.2.2 for more on additive inverses.

Adding polynomials isn't difficult at all. The only problem is that you can only add certain parts of each polynomial together.

The Opposite of a Polynomial

The **opposite** of a number is its **additive inverse**. The opposite of a positive number is its corresponding negative number, and vice versa.

For example, -1 is the opposite of 1 , and 1 is the opposite of -1 .

To find the opposite of a polynomial, you make the **positive terms negative** and the **negative terms positive**.

Example 1

Find the opposites of the following polynomials:

a) $2x - 1$ b) $-5x^2 + 3x - 1$

Solution

a) $-2x + 1$ b) $5x^2 - 3x + 1$

✓ Guided Practice

Find the opposites of the following polynomials.

1. $2x + 1$

2. $-5x - 1$

3. $x^2 + 5x - 2$

4. $3x^2 - 2x + 3$

5. $3x^2 + 4x - 8$

6. $-8x^2 - 4x + 4$

7. $4x^4 - 16$

8. $8x^3 - 6x^2 + 6x - 8$

9. $5x^4 - 6x^2 + 7$

10. $-2x^4 + 3x^3 - 2x^2$

11. $-0.9x^3 - 0.8x^2 - 0.4x - 1.0$

12. $-1.4x^3 - 0.8x^2 - \frac{1}{2}x$

Adding Polynomials

Adding polynomials consists of **combining** all **like terms**.

There are a few ways of adding polynomials — two of the methods are shown in Example 2.

Example 2

Find the sum of $-5x^2 + 3x - 1$, $6x^2 - x + 3$, and $5x - 7$.

Solution

Method A — Collecting Like Terms and Simplifying

$$\begin{aligned} &(-5x^2 + 3x - 1) + (6x^2 - x + 3) + (5x - 7) \\ &= -5x^2 + 3x - 1 + 6x^2 - x + 3 + 5x - 7 \\ &= \underbrace{-5x^2 + 6x^2} + \underbrace{3x - x + 5x} + \underbrace{-1 + 3 - 7} \\ &= x^2 + 7x - 5 \end{aligned}$$

Method B — Vertical Lining Up of Terms

$$\begin{array}{r} -5x^2 + 3x - 1 \\ + \quad 6x^2 - x + 3 \\ + \quad \quad 5x - 7 \\ \hline x^2 + 7x - 5 \end{array}$$

Both methods give the same solution.

Don't forget:

See Topic 1.2.4 for the definition of multiplication.

Multiplying a Polynomial by a Number

Multiplying a polynomial by a number is the same as adding the polynomial together several times.

Example 3

Multiply $x + 3$ by 3.

Solution

$$\begin{aligned} (x + 3) \times 3 &= (x + 3) + (x + 3) + (x + 3) \\ &= x + x + x + 3 + 3 + 3 \\ &= 3x + 9 \end{aligned}$$

Don't forget:

See Section 1.2 for more about the distributive property.

The simple way to do this is to **multiply each term** of the polynomial **by the number**. In other words, you **multiply out the parentheses**, using the distributive property of multiplication over addition.

Example 4

Multiply $x^2 + 2x - 4$ by 3.

Solution

$$\begin{aligned} 3(x^2 + 2x - 4) &= (3 \times x^2) + (3 \times 2x) - (3 \times 4) \\ &= \mathbf{3x^2 + 6x - 12} \end{aligned}$$

✓ Guided Practice

Add these polynomials and simplify the resulting expressions.

13. $(4x^2 - 2x - 1) + (3x^2 + x - 10)$

14. $(11x^4 - 5x^3 - 2x) + (-7x^4 + 3x^3 + 5x - 3)$

15. $-5c^3 - 3c^2 + 2c + 1$

16. $5x^2 + 3x - 3$

$$4c^2 - c - 3$$

$$-4x^2 - 3x + 5$$

$$\underline{6c^3 \quad + \quad c + 4}$$

$$\underline{-2x^2 + x - 7}$$

Multiply these polynomials by 4.

17. $10y^2 - 7y + 5$

18. $(x^2 - 3x + 3)$

19. $(-x^2 + x - 4)$

20. $(2x^2 + 5x + 2)$

✓ Independent Practice

In Exercises 1-7, simplify the expression and state the degree of the resulting polynomial.

1. $(2x^2 + 3x - 7) + (7x^2 - 3x + 4)$

2. $(x^3 + x - 4) + (x^3 - 8) + (4x^3 - 3x - 1)$

3. $(-x^6 + x - 5) + (2x^6 - 4x - 6) + (-2x^6 + 2x - 4)$

4. $(3x^2 - 2x + 7) + (4x^2 + 6x - 8) + (-5x^2 + 4x - 5)$

5. $(0.4x^3 - 1.1) + (0.3x^3 + x - 1.0) + (1.1x^3 + 2.1x - 2.0)$

6. $-4a^3 - 2a + 3$

7. $1.1c^2 + 1.4c - 0.48$

$$8a^4 - 2a^3 - 4a + 8$$

$$-4.9c^2 - 3.6c + 0.98$$

$$\underline{7a^4 \quad - \quad 4a - 7}$$

$$\underline{7.3c^2 \quad + \quad 0.13}$$

Multiply each polynomial below by -4 .

8. $4a^2 + 3a - 2$

9. $-c^2 + 3c + 1$

10. $-6x^3 - 4x^2 + x - 8$

11. $24x^3 + 16x^2 - 4x + 32$

Multiply each polynomial below by $2a$.

12. $3a^2 + a - 8$

13. $-7a^4 + 2a^2 - 5a + 4$

Round Up

Adding polynomials can look hard because there can be several terms in each polynomial. The important thing is to combine each set of like terms, step by step.

Subtracting Polynomials

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

10.0: Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

What it means for you:

You'll learn how to subtract polynomials.

Key words:

- polynomial
- like terms
- inverse

Subtracting one polynomial from another follows the same rules as adding polynomials — you just need to combine like terms, then carry out all the subtractions to simplify the expression.

Subtracting Polynomials

Subtracting polynomials is the same as subtracting numbers.

To subtract Polynomial A from Polynomial B, you need to **subtract each term** of Polynomial A from Polynomial B. Then you can combine any like terms to simplify the expression.

Example 1

Subtract Polynomial A from Polynomial B, where Polynomial A = $x^2 + x$ and Polynomial B = $x^2 + 4x$.

Solution

Subtract each term of Polynomial A from Polynomial B:

$$\begin{aligned} \text{Polynomial B} - \text{Polynomial A} &= x^2 + 4x - (x^2) - (x) \\ &= x^2 - x^2 + 4x - x \\ &= 0 + 3x \\ &= 3x \end{aligned}$$

Guided Practice

1. Subtract $x^2 - 4$ from $x^2 + 8$.
2. Subtract $3x - 4$ from $8x^2 - 5x + 4$.
3. Subtract $x + 4$ from $x^2 - x$.
4. Subtract $x^2 - 16$ from $x^2 + 8$.
5. Subtract $x^2 + x - 1$ from $x + 4$.
6. Subtract $-3x^2 + 4x - 5$ from $x^2 - 7$.
7. Subtract $-3x^2 - 5x + 2$ from $-2x^3 - x^2 - 7x$.

Simplify:

8. $(9a - 10) - (5a + 2)$
9. $(5a^2 - 2a + 3) - (3a + 5)$
10. $(x^3 + 5x^2 - x) - (x^2 + x)$

Subtracting is Simply Adding the Opposite

Don't forget:

See Topic 1.2.5 for the definition of subtraction.

Another way to look at subtraction of polynomials is to go back to the definition of subtraction.

When you **subtract Polynomial A** from Polynomial B, what you're actually doing is **adding the opposite of Polynomial A** to Polynomial B.

Example 2

Subtract $-5x^2 + 3x - 8$ from $-7x^2 + x + 5$.

Solution

$$\begin{aligned} & -7x^2 + x + 5 - (-5x^2 + 3x - 8) \\ &= -7x^2 + x + 5 + 5x^2 - 3x + 8 \\ &= -7x^2 + 5x^2 + x - 3x + 5 + 8 \\ &= \mathbf{-2x^2 - 2x + 13} \end{aligned}$$

Alternatively, you can do subtraction by lining up terms vertically:

Example 3

Subtract $-5x^2 + 3x - 8$ from $-7x^2 + x + 5$.

Solution

$$\begin{array}{r} -7x^2 + x + 5 \\ - (-5x^2 + 3x - 8) \\ \hline -2x^2 - 2x + 13 \end{array} \quad \text{OR} \quad \begin{array}{r} -7x^2 + x + 5 \\ + (5x^2 - 3x + 8) \\ \hline -2x^2 - 2x + 13 \end{array}$$

This is the opposite
of $-5x^2 + 3x - 8$

✓ Guided Practice

Simplify the expressions in Exercises 11–16.

11. $(3a^4 + 4) - (2a^2 - 5a^4)$

12. $(6x^2 + 8 - 9x^4) - (3x - 4 + x^3)$

13. $(9c^2 + 11c^2 + 5c - 5) - (-10 + 4c^4 - 8c + 3c^2)$

14. $(8a^2 - 2a + 5a) - (9a^2 + 2a + 4)$

15. $\begin{array}{r} 6x^2 - 6 \\ -(5x^2 + 9) \end{array}$

16. $\begin{array}{r} 8a^2 + 4a - 9 \\ -(3a^2 - 3a + 7) \end{array}$

17. Subtract $7a^3 + 3a - 12$ from $5a^2 - a + 4$ by adding the opposite expression. Use the vertical lining up method.

18. Subtract $(8p^3 - 11p^2 - 3p)$ from $4p^3 + 6p^2 - 10$ by adding the opposite expression. Use the vertical lining up method.

If you add a polynomial to its opposite, the result will always be 0.

Example 4

Find the sum of $-5x^2 + 3x - 1$ and $5x^2 - 3x + 1$.

Solution

$$\begin{aligned} & -5x^2 + 3x - 1 + (5x^2 - 3x + 1) \\ &= -5x^2 + 3x - 1 + 5x^2 - 3x + 1 \\ &= \underbrace{-5x^2 + 5x^2} + \underbrace{3x - 3x} + \underbrace{-1 + 1} \\ &= 0 + 0 + 0 \\ &= 0 \end{aligned}$$

Check it out:

You can use this method to check answers to “opposites” questions.

Independent Practice

Subtract the polynomials and simplify the resulting expression.

- $(5a + 8) - (3a + 2)$
- $(8x - 2y) - (8x + 4y)$
- $(-4x^2 + 7x - 3) - (2x^2 - 4x + 6)$
- $(3a^2 + 2a + 6) - (2a^2 + a + 3)$
- $-3x^4 - 2x^3 + 4x - 1 - (-2x^4 - x^3 + 3x^2 - 5x + 3)$
- $5 - [(2k + 3) - (3k + 1)]$
- $-10a^2 + 4a - 1$
 $- (7a^2 + 4a)$
- $(x^2 + 4x + 6)$
 $- (2x^2 + 2x + 4)$

Solve these by first simplifying the left side of the equations.

- $(2x + 3) - (x - 7) = 40$
- $(4x + 14) - (-10x - 3) = 73$
- $(2 - 3x) - (7 - 2x) = 23$
- $(17 - 5x) - (4 - 3x) - (6 - x) = 19$

Find the opposite of the polynomials below.

- $x^2 + 2x + 1$
- $-a^2 + 6a + 4$
- $4b^2 - 6bc + 7c$
- $a^3 + 4a^2 + 3a - 2$

- The opposite of a fifth degree polynomial has what degree?
- If a monomial is subtracted from another monomial, what are the possible results?
- What is the degree of the polynomial formed when a 2nd degree polynomial is subtracted from a 1st degree polynomial?
- A 3rd degree polynomial has a 2nd degree polynomial subtracted from it. What is the degree of the resulting polynomial?

Round Up

Watch out for the *signs* when you're subtracting polynomials. It's usually a good idea to put *parentheses* around the polynomial you're subtracting, to make it easier to keep track of the *signs*.

Adding and Subtracting Polynomials

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

10.0: Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

What it means for you:

You'll use addition and subtraction of polynomials to solve more complicated problems.

Key words:

- polynomial
- like terms

Don't forget:

Remember that the perimeter of a rectangle is the sum of the lengths of all four sides.

Don't forget:

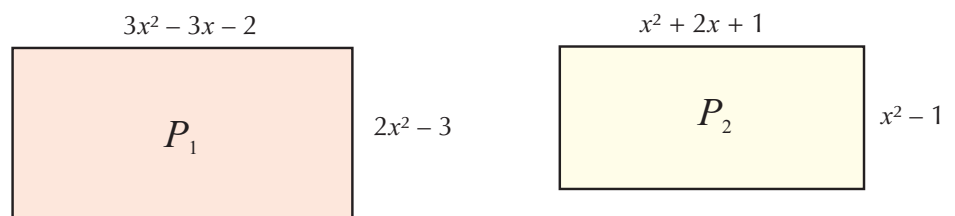
Be really careful in part b) — the minus sign applies to both terms inside the parentheses.

This Topic just contains one big example that uses both addition and subtraction of polynomials.

An Example Using Addition and Subtraction

Example 1

Find a) the sum of and b) the difference between the perimeters of the rectangles shown below:



Solution

The first thing you need to do is find and simplify expressions for each of the perimeters.

$$\begin{aligned} \text{Perimeter of } P_1 &= (3x^2 - 3x - 2) + (3x^2 - 3x - 2) + (2x^2 - 3) + (2x^2 - 3) \\ &= 3x^2 - 3x - 2 + 3x^2 - 3x - 2 + 2x^2 - 3 + 2x^2 - 3 \\ &= 3x^2 + 3x^2 + 2x^2 + 2x^2 - 3x - 3x - 2 - 2 - 3 - 3 \\ &= \mathbf{10x^2 - 6x - 10} \end{aligned}$$

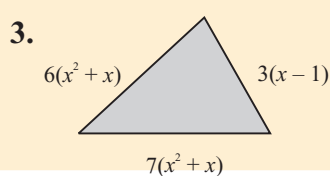
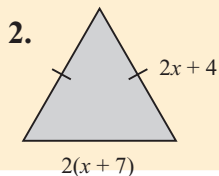
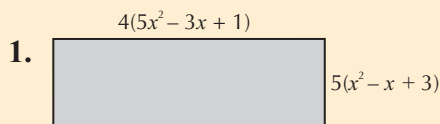
$$\begin{aligned} \text{Perimeter of } P_2 &= (x^2 + 2x + 1) + (x^2 + 2x + 1) + (x^2 - 1) + (x^2 - 1) \\ &= x^2 + 2x + 1 + x^2 + 2x + 1 + x^2 - 1 + x^2 - 1 \\ &= x^2 + x^2 + x^2 + x^2 + 2x + 2x + 1 + 1 - 1 - 1 \\ &= \mathbf{4x^2 + 4x} \end{aligned}$$

$$\begin{aligned} \text{a) Perimeter of } P_1 + \text{Perimeter of } P_2 &= (10x^2 - 6x - 10) + (4x^2 + 4x) \\ &= 10x^2 - 6x - 10 + 4x^2 + 4x \\ &= 10x^2 + 4x^2 - 6x + 4x - 10 \\ &= \mathbf{14x^2 - 2x - 10} \end{aligned}$$

$$\begin{aligned} \text{b) Perimeter of } P_1 - \text{Perimeter of } P_2 &= (10x^2 - 6x - 10) - (4x^2 + 4x) \\ &= 10x^2 - 6x - 10 - 4x^2 - 4x \\ &= 10x^2 - 4x^2 - 6x - 4x - 10 \\ &= \mathbf{6x^2 - 10x - 10} \end{aligned}$$

✓ Guided Practice

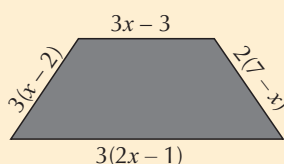
Find the perimeter of each of these shapes.



✓ Independent Practice

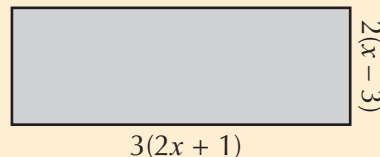
Simplify each of these expressions.

- $4y - [3(y - x) - 5(x + y)]$
- $3[2y - 3(y - 1) + 2(-y - 1)]$
- $3(x - y) - 2(x - 3y)$
- $-2[(x - y) - 2(x + 2y)] - [(-2x - y) - (x + 2y)]$
- $(-3x^3 - x^2y + 2xy^2 - 2y^3) - (-5x^3 + x^2y - 3xy^2 - 3y^3) + (-x^3 - 2x^2y + y^3)$



6. Find the difference between the perimeters of the trapezoid and the triangle shown.

- Simplify $5(4x + 3) - 4(3x - 1) + 3(x - 1)$.
- Simplify $(8x^3 - 5x^2 - 2x - 7) - (6x^3 - 3x^2 + x - 5) - (-3x^2 + 3x - 2)$.
- Simplify $-2(3x^3 - 2x) - 3(-x^3 + 7) - (2x^2 - 5x^3 - 5)$.
- Find the sum of the opposites of:
 $-2x^3 + 3x^2 - 5x + 1$ and $3x^3 - 2x^2 + 3x - 3$
- Find the difference between the opposites of:
 $-2x^2 - 3x + 5$ and $3x^2 + 2x - 4$
- Simplify $-2(3x^{2p} - 2x^p + 1) + 4(2x^{2p} - x^p - 3) - 5(x^{2p} - 2x^p - 1)$.
- If the perimeter of the figure shown is 90 inches, what are the dimensions of the figure?



Round Up

You've covered addition and subtraction of polynomials in just four short Topics. As you might expect, the next Section covers *multiplication* of polynomials, and then you'll learn about *division* of polynomials in Section 6.3.

Topic 6.2.1

Section 6.2 Rules of Exponents

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

10.0: Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

What it means for you:

You'll multiply and divide algebraic expressions using the rules of exponents.

Key words:

• exponent

Don't forget:

See Topic 1.3.1 for more on the rules of exponents.

Don't forget:

Rules 3 and 4 mean that here you multiply each power from inside the parentheses by 2, the power **outside** the parentheses.

You learned about the rules of exponents in Topic 1.3.1 — in this Topic, you'll apply those same rules to *monomials* and *polynomials*.

Use the Rules of Exponents to Simplify Expressions

These are the same rules you learned in Chapter 1, but this time you'll use them to simplify algebraic expressions:

Rules of Exponents

- 1) $x^a \cdot x^b = x^{a+b}$
- 2) $x^a \div x^b = x^{a-b}$ (if $x \neq 0$)
- 3) $(x^a)^b = x^{ab}$
- 4) $(cx)^b = c^b x^b$
- 5) $x^{\frac{a}{b}} = \sqrt[b]{x^a}$ or $(\sqrt[b]{x})^a$
- 6) $x^{-a} = \frac{1}{x^a}$ (if $x \neq 0$)
- 7) $x^0 = 1$

Example 1

Simplify the expression $(-2x^2m)(-3x^3m^3)$.

Solution

$$\begin{aligned} & (-2x^2m)(-3x^3m^3) \\ &= (-2)(-3)(x^2)(x^3)(m)(m^3) && \text{Put all like variables together} \\ &= 6x^{2+3} \cdot m^{1+3} && \text{Use Rule 1 and add the powers} \\ &= 6x^5m^4 \end{aligned}$$

Example 2

Simplify the expression $(3a^2xb^3)^2$.

Solution

$$\begin{aligned} & (3a^2xb^3)^2 \\ &= 3^2 \cdot a^{2 \cdot 2} \cdot x^2 \cdot b^{3 \cdot 2} && \text{Use Rules 3 and 4} \\ &= 9a^4x^2b^6 \end{aligned}$$

Example 3

Simplify the expression $\frac{-10x^5m^3v^2}{-5x^4mv^2}$.

Solution

$$\begin{aligned} & \frac{-10x^5m^3v^2}{-5x^4mv^2} \\ &= \frac{-10}{-5} \cdot \frac{x^5}{x^4} \cdot \frac{m^3}{m} \cdot \frac{v^2}{v^2} \\ &= 2x^1m^2v^0 \\ &= \mathbf{2xm^2} \end{aligned}$$

Separate the expression into parts that have only one variable

Use Rule 2 and subtract the powers

From Rule 5, anything to the power 0 is just 1

Guided Practice

Simplify each expression.

1. $-3at(4a^2t^3)$

2. $(-5x^3yt^2)(-2x^2y^3t)$

3. $(-2x^2y^3)^3$

4. $-2mx(3m^2x - 4m^2x + m^3x^3)$

5. $(-3x^2t)^3(-2x^3t^2)^2$

6. $-2mc(-3m^2c^3 + 5mc)$

7. $\frac{10m^3n^8z^6}{2m^2n^3z}$

8. $\frac{14a^2b^4c^8}{4a^7b^4}$

9. $\frac{12j^8k^{-8}m^{-1}}{8j^2k^{-10}m^4}$

10. $\frac{16b^9a^4j^4}{32(b^5a^3c)^2}$

Independent Practice

Simplify.

1. $\left(\frac{1}{2}a^3b^4\right)^0$

2. $\left(\frac{2}{8}xy^4z^3\right)^{-2}$

3. $4a^2(a^2 - b^2)$

4. $4m^2x^2(x^2 + x + 1)$

5. $a(a + 4) + 4(a + 4)$

6. $2a(a - 4) - 3(a - 4)$

7. $m^2n^3(mx^2 + 3nx + 2) - 4m^2n^3$

8. $4m^2n^2(m^3n^8 + 4) - 3m^3n^{10}(m^2 + 2n^3)$

9. $\frac{2^5h^6y^2}{2^2h^2y^7}$

10. $\frac{10b^8a^{17}}{2b^6a^{19}}$

11. $\frac{2b^4c^5}{6b^3c^6}$

12. $\frac{4a^3m^4}{20a^7m^6}$

Find the value of ? that makes these statements true.

13. $m^2(m^4 + 2m^3) = m^6 + 2m^5$

14. $m^4a^6(3m^2a^8 + 4m^2a^7) = 3m^7a^{14} + 4m^6a^9$

15. $\frac{4x^4y^3}{2x^7y^2} = \frac{2}{x^3y}$

16. $\frac{4x^2y^8c^7}{18x^{10}y^2c^6} = \frac{2yc}{9x^3}$

Round Up

You can apply the rules of exponents to *any algebraic values*. In this Topic you just dealt with *monomials*, but the rules work with expressions with *more than one term* too.

Polynomial Multiplication

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power.

They understand and use the rules of exponents.

10.0: Students add, subtract, multiply, and divide monomials and

polynomials. Students solve multistep problems, including word problems, by using these techniques.

What it means for you:

You'll learn how to multiply monomials and polynomials.

Key words:

- polynomial
- monomial
- distributive property
- degree

To multiply two polynomials together, you have to multiply every single term together, one by one.

The Distributive Property and Polynomial Products

In Topic 6.1.2 you saw the method for **multiplying a polynomial by a number** — you **multiply each term separately by that number**.

This method's based on the distributive property from Topic 1.2.7.

In the same sort of way, when you **multiply a polynomial by a monomial**, you **multiply each term separately by that monomial** — again, using the distributive property.

Example 1

Simplify the expression $-2a(a + 3a^2)$.

Solution

$-2a(a + 3a^2)$ is a product of the monomial $-2a$ and the binomial $(a + 3a^2)$, so multiply each term of the binomial by the monomial.

$$\begin{aligned} &= -2a(a) + (-2a)(3a^2) \\ &= -2a^2 - 6a^3 \end{aligned}$$

To find the product of **two polynomials**, such as $(a - 2b)(3a + b)$, you use the **distributive property twice**:

Example 2

Simplify the expression $(a - 2b)(3a + b)$.

Solution

$$\begin{aligned} &(a - 2b)(3a + b) \\ &= (a)(3a + b) + (-2b)(3a + b) \\ &= [(a)(3a) + (a)(b)] + [(-2b)(3a) + (-2b)(b)] \\ &= (3a^2 + ab) + (-6ab - 2b^2) \\ &= 3a^2 + ab - 6ab - 2b^2 \\ &= 3a^2 - 5ab - 2b^2 \end{aligned}$$

Use the distributive property twice

Example 3Simplify $(3x - 2m)(4x - 3m)$.**Solution**

$$\begin{aligned}
 &(3x - 2m)(4x - 3m) \\
 &= 3x(4x - 3m) - 2m(4x - 3m) \\
 &= 12x^2 - 9mx - 8mx + 6m^2 \\
 &= \mathbf{12x^2 - 17mx + 6m^2}
 \end{aligned}$$

Example 4Simplify $(v + 3)(4 + v)$.**Solution**

$$\begin{aligned}
 &(v + 3)(4 + v) \\
 &= v(4 + v) + 3(4 + v) \\
 &= 4v + v^2 + 12 + 3v \\
 &= \mathbf{v^2 + 7v + 12}
 \end{aligned}$$

 **Guided Practice**

Expand and simplify each product, using the distributive method. Show all your work.

1. $(m + c)(m + 2c)$

2. $(x - 3y)(x + 2y)$

3. $(2x - 3)(2x + 5)$

4. $(a - 4b)(a + 3b)$

5. $(3x - 5)(2x - 3)$

6. $(5x + 3y)(2x + 3y)$

Determine whether the following are correct for the products given.

7. $(a + b)(a - b) = a^2 - b^2$

8. $(a + b)^2 = a^2 + b^2$

9. $(a - b)^2 = a^2 - 2ab + b^2$

10. $(a + b)(a + b) = a^2 + 2ab + b^2$

You Can Multiply Polynomials with Lots of Terms**Example 5**Simplify $(x + 2)(x^2 + 2x + 3)$.**Solution**

$$\begin{aligned}
 &(x + 2)(x^2 + 2x + 3) \\
 &= x(x^2 + 2x + 3) + 2(x^2 + 2x + 3) \\
 &= x^3 + 2x^2 + 3x + 2x^2 + 4x + 6 \\
 &= \mathbf{x^3 + 4x^2 + 7x + 6}
 \end{aligned}$$

Check it out:

Multiply each term in one set of parentheses by every term in the second set of parentheses.

The Highest Power Gives the Degree of a Polynomial

Example 6

Simplify $(x - 3)(2x^2 - 3x + 2)$ and state the degree of the product.

Solution

$$\begin{aligned}(x - 3)(2x^2 - 3x + 2) &= x(2x^2 - 3x + 2) - 3(2x^2 - 3x + 2) \\ &= 2x^3 - 3x^2 + 2x - 6x^2 + 9x - 6 \\ &= \mathbf{2x^3 - 9x^2 + 11x - 6}\end{aligned}$$

The term $2x^3$ has the highest power, so the degree is 3.

Guided Practice

Expand and simplify each product, and state the degree of the resulting polynomial.

11. $(x + 3)(2x^2 - 3x + 1)$

12. $(2y - 3)(-3y^2 - y + 1)$

13. $(x^2 - 3x + 4)(2x + 1)$

14. $(3y^3 + 4y - 2)(4y - 1)$

15. $(3x + 4)(-2x^2 + x - 2)$

16. $(2x - 3)^2$

Determine whether the following are correct for the products given.

17. $(a^2 + b^2)(a - ab + b) = a^3 + b^3$ 18. $(a + b)(a^2 - ab + b^2) = a^3 + b^3$

19. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$ 20. $(a^2 - b^2)(a + ab + b) = a^3 - b^3$

You Can Also Use the Stacking Method

You can find the product of 63 and 27 by “stacking” the two numbers and doing long multiplication:

$$\begin{array}{r} 63 \\ \times 27 \\ \hline 441 \\ + 1260 \\ \hline 1701 \end{array}$$

$\swarrow 7 \times 63$
 $\swarrow 2 \times 63$

Don't forget:

The units, tens, hundreds, and thousands are in separate columns.

You can use the same idea to find the products of polynomials — just make sure you keep like terms in the same columns.

Example 7

Expand and simplify the product $(2x + 3y)(x + 5y)$.

Solution

$$\begin{array}{r} 2x + 3y \\ \times x + 5y \\ \hline 10xy + 15y^2 \\ + 2x^2 + 3xy \\ \hline \mathbf{2x^2 + 13xy + 15y^2} \end{array}$$

$\swarrow 5y(2x + 3y)$
 $\swarrow x(2x + 3y)$

Don't forget:

Keep like terms in the same column.

Example 8Simplify $(x - 2)(2x^2 - 3x + 4)$.**Solution**

$$\begin{array}{r}
 2x^2 - 3x + 4 \\
 \times \quad x - 2 \\
 \hline
 -4x^2 + 6x - 8 \\
 2x^3 - 3x^2 + 4x \\
 \hline
 \mathbf{2x^3 - 7x^2 + 10x - 8}
 \end{array}$$

$\swarrow -2(2x^2 - 3x + 4)$
 $\swarrow x(2x^2 - 3x + 4)$

Guided Practice

Use the stacking method to multiply these polynomials:

21. $(3x + y)(x + 2y)$

22. $(4x + 5y)(2x + 3y)$

23. $(3x^2 + 2x + 3)(3x - 4)$

24. $(4x^2 - 5x + 6)(4x + 5)$

25. $(a + b)^2$

26. $(a - b)^2$

27. $(a - b)(a + b)$

28. $(a - b)(a^2 + ab + b^2)$

29. $(a + b)(a^2 - ab + b^2)$

30. $(a^2 - b^2)(a^2 + b^2)$

Independent Practice

Expand and simplify each product, using the distributive method. Show all your work.

1. $(2x + 8)(x - 4)$

2. $(x^2 + 3)(x - 2)$

3. $(x - 3)(2 - x)$

4. $(2x + 7)(3x + 5)$

5. $(3x - 8)(x^2 - 4x + 2)$

6. $(2x - 4y)(3x - 3y + 4)$

Use the stack method to multiply. Show all your work.

7. $(x^2 - 4)(x + 3)$

8. $(x - y)(3x^2 + xy + y^2)$

9. $(4x^2 - 5x)(1 + 2x - 3x^2)$

10. $(x + 4)(3x^2 - 2x + 5)$

Use these formulas to find each of the products in Exercises 11–16.

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

11. $(x + 2)^2$

12. $(3x - 1)(3x + 1)$

13. $(2x - 3)^2$

14. $(4x + y)^2$

15. $(5x + 3c)(5x - 3c)$

16. $(8c + 3)^2$

Round Up

You've seen the *distributive property* lots of times already, but it's easy to make calculation errors when you're multiplying *long polynomials* — so be careful and *check your work*.

Polynomial Multiplication — Area and Volume

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power.

They understand and use the rules of exponents.

10.0: Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems,

including word problems, by using these techniques.

What it means for you:

You'll multiply polynomials to solve problems involving area and volume.

Key words:

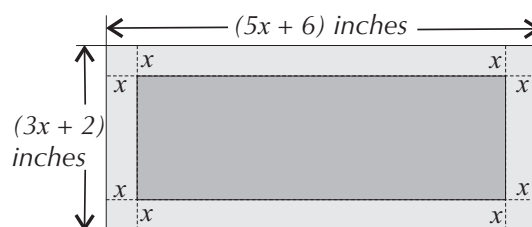
- polynomial
- monomial
- distributive property

Polynomial multiplication isn't just about abstract math problems. Like everything in math, you can use it to work out problems dealing with everyday life.

Find Areas by Multiplying Polynomials

Example 1

Find the area of the space between the two rectangles:



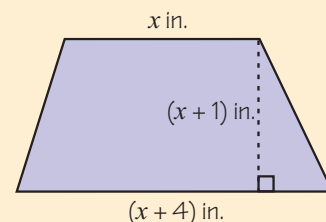
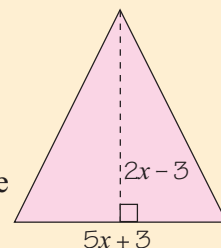
Solution

The length of the middle rectangle is $5x + 6 - 2x = (3x + 6)$ inches.
The width of the middle rectangle is $3x + 2 - 2x = (x + 2)$ inches.

$$\begin{aligned} \text{Area of space} &= \text{area of large rectangle} - \text{area of small rectangle} \\ &= (5x + 6)(3x + 2) - (3x + 6)(x + 2) \\ &= 15x^2 + 10x + 18x + 12 - (3x^2 + 6x + 6x + 12) \\ &= 15x^2 + 28x + 12 - 3x^2 - 12x - 12 \\ &= \mathbf{(12x^2 + 16x) \text{ in}^2} \end{aligned}$$

✓ Guided Practice

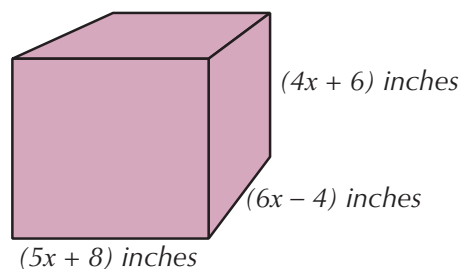
- Find the area of a rectangle whose dimensions are $(3x + 4)$ inches by $(2x + 1)$ inches.
- Find the area of the triangle shown on the right. (The formula for the area of a triangle is $\text{Area} = \frac{1}{2}bh$.)
- The height of a triangle is $(3x - 2)$ inches and its base is $(4x + 10)$ inches. Find the area of the triangle.
- Find the area of a rectangle whose dimensions are $(3 + 2x)$ inches by $(5 + 6x)$ inches.
- Find the area of a square with side length $(a^2 + b^2 - c^2)$ feet.
- The area of a trapezoid is given by $A = \frac{1}{2}h(b_1 + b_2)$ where h is the height and b_1 and b_2 are the lengths of the parallel sides. Find the area of this trapezoid.



Multiply Polynomials to Find Volumes

Example 2

Find the volume of the box on the right.



Solution

$$\text{Volume} = \text{Length} \times \text{Width} \times \text{Height}$$

$$= (5x + 8)(6x - 4)(4x + 6)$$

$$= [5x(6x - 4) + 8(6x - 4)](4x + 6) \quad \text{Multiply the first two polynomials}$$

$$= (30x^2 - 20x + 48x - 32)(4x + 6)$$

$$= (30x^2 + 28x - 32)(4x + 6) \quad \text{Simplify the first product}$$

$$= 4x(30x^2 + 28x - 32) + 6(30x^2 + 28x - 32) \quad \text{Multiply out again}$$

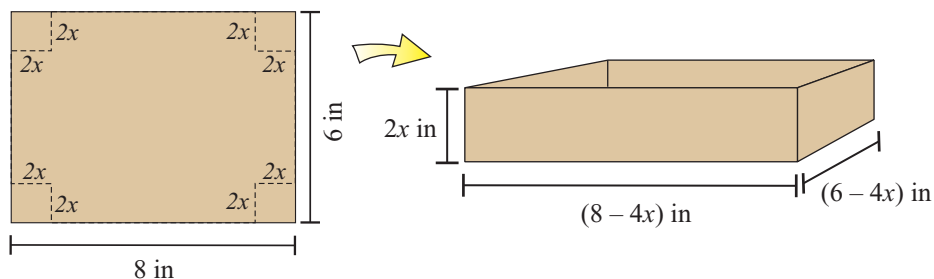
$$= 120x^3 + 112x^2 - 128x + 180x^2 + 168x - 192$$

$$= 120x^3 + 112x^2 + 180x^2 - 128x + 168x - 192 \quad \text{Commutative law}$$

$$= (120x^3 + 292x^2 + 40x - 192) \text{ in}^3$$

Example 3

Find the volume of a box made from the sheet below by removing the four corners and folding.



Solution

$$\text{Volume} = \text{Length} \times \text{Width} \times \text{Height}$$

$$= (8 - 4x)(6 - 4x)(2x)$$

$$= [8(6 - 4x) - 4x(6 - 4x)](2x) \quad \text{Multiply the first two polynomials}$$

$$= (48 - 32x - 24x + 16x^2)(2x)$$

$$= (48 - 56x + 16x^2)2x \quad \text{Simplify the first product}$$

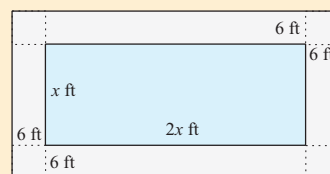
$$= 96x - 112x^2 + 32x^3 \quad \text{Multiply by the third polynomial}$$

$$= (32x^3 - 112x^2 + 96x) \text{ in}^3$$

✓ Guided Practice

7. Find the volume of a cube with side length $(3x + 6)$ inches.

8. A concrete walkway around a swimming pool is 6 feet wide. If the length of the pool is twice the width, x feet, what is the combined area of the walkway and pool?

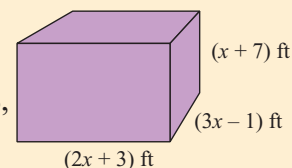


Use the rectangular prism shown to answer these questions.

9. Find the volume of the prism.

10. Find the surface area of the prism.

11. If the height of the prism was reduced by 10%, what would be the new volume of the prism?



✓ Independent Practice

Expand and simplify the following.

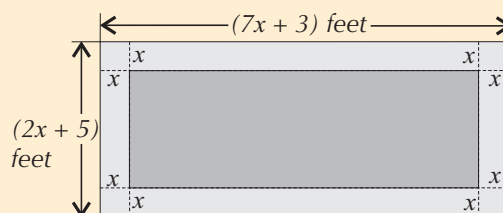
1. $(3y + 5)^3$

2. $(2y - 1)^3$

3. The area of a parallelogram is given by the formula $A = bh$, where b is the length of the base and h is the height of the parallelogram.

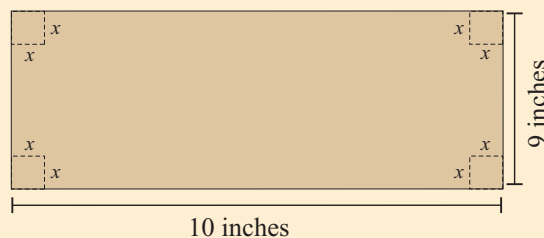
Find the area of a parallelogram that has a base length of $(2x^2 + 3x - 1)$ cm and a height of $(3x - 1)$ cm.

4. A gardener wants to put a walkway around her garden, as shown on the right. What is the area of the walkway?



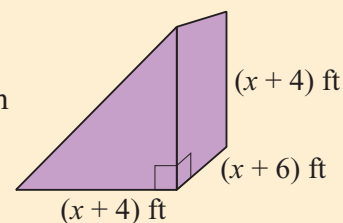
5. Obike made a box from a 10 inch by 9 inch piece of cardboard by cutting squares of x units from each of the four corners.

Find the volume of his box.



6. Find the volume of the solid shown.

7. Find the volume of another triangular prism that has the same base measurements as this one but a height 25% less than the height shown here.



Round Up

For problems involving *area*, you'll have to *multiply two terms or polynomials together*.

For problems involving *volume*, you'll have to *multiply three terms or polynomials*.

Topic 6.3.1

Section 6.3 Division by Monomials

California Standards:

10.0: Students add, subtract, multiply, and **divide monomials and polynomials**. Students solve multistep problems, including word problems, by using these techniques.

What it means for you:

You'll learn how to use the rules of exponents to divide a polynomial by a monomial.

Key words:

- polynomial
- monomial
- exponent
- distributive property

Don't forget:

You covered the rules of exponents in Topics 1.3.1 and 6.2.1.

Don't forget:

Like with multiplying, you need to treat the powers of x and y separately.

Those *rules of exponents* that you saw in Topic 6.2.1 really are useful. In this Topic you'll use them to *divide polynomials by monomials*.

Dividing a Polynomial by a Monomial

To **divide** a polynomial **by a monomial**, you need to use the **rules of exponents**.

The particular rule that's useful here is:

$$\frac{x^a}{x^b} = x^{a-b} \text{ provided } x \neq 0$$

Example 1

Divide $2x^2$ by x .

Solution

$$\frac{2x^2}{x} = 2x^{2-1} = 2x^1 = 2x$$

Example 2

Divide $2x^3y + xy^2$ by xy .

Solution

$$\begin{aligned} \frac{2x^3y + xy^2}{xy} &= \left(\frac{2x^3y}{xy} \right) + \left(\frac{xy^2}{xy} \right) \\ &= \left(\frac{2x^3}{x} \cdot \frac{y}{y} \right) + \left(\frac{x}{x} \cdot \frac{y^2}{y} \right) \\ &= (2x^{3-1} \cdot y^{1-1}) + (x^{1-1} \cdot y^{2-1}) \\ &= (2x^2 \cdot 1) + (1 \cdot y^1) \\ &= 2x^2 + y \end{aligned}$$

Divide each term in the expression by xy , using the distributive property

Example 3

Simplify $\frac{2m^3c^2v - 4m^2c^3v + 10mc^4v^3}{-2mc^2v}$.

Solution

$$\begin{aligned} & \frac{2m^3c^2v - 4m^2c^3v + 10mc^4v^3}{-2mc^2v} \\ &= \left(\frac{\cancel{2}m^3\cancel{c^2}\cancel{v}}{\cancel{-2}m\cancel{c^2}\cancel{v}} \right) - \left(\frac{-4\cancel{m}^2\cancel{c}^3\cancel{v}}{\cancel{-2}m\cancel{c}^2\cancel{v}} \right) + \left(\frac{10\cancel{m}c^4v^3}{\cancel{-2}m\cancel{c}^2v} \right) \\ &= (-1 \cdot m^{3-1}) - (-2 \cdot m^{2-1} \cdot c^{3-2}) + (-5 \cdot c^{4-2} \cdot v^{3-1}) \\ &= \mathbf{-m^2 + 2mc - 5c^2v^2} \end{aligned}$$

Don't forget:

You can cancel anything that is on both the top and the bottom of the fraction. See Topic 6.3.3.

Guided Practice

Simplify each of these quotients.

- $9m^3c^2v^4 \div (-3m^2cv^3)$
- $\frac{-12x^5y^6z^4}{-4x^3y^5z^2}$
- $\frac{-10m^3x^2 + 15m^4x^3}{-5m^3x^2}$
- $\frac{6x^4y^5t^3 - 12x^3y^4t^2 - 3x^5y^3t^3}{-3x^3y^3t^2}$
- $-\frac{14x^5y^8a^4z^{12}}{7x^4y^3z^9}$
- $\frac{8a^9df^9k^3 - 12a^8d^6f^5k^3 + 28ca^8d^8k^4}{-4a^5dk^2}$

Independent Practice

Simplify each of these quotients.

- $\frac{8x^3 - 4x^2 + 12x}{-4x}$
- $\frac{6x^4 - 3x^3 + 9x^2 - 15x}{3x}$
- $\frac{-12m^3c^2v^4 + 4m^2c^3v^3 - 16m^2c^2v^2}{-4m^2c^2v^2}$
- $\frac{4xy^2z^3 + 8x^2y^3z^4}{xyz^2}$
- Divide $15x^5 - 10x^3 + 25x^2$ by $-5x^2$.
- Divide $20a^6b^4 - 14a^7b^5 + 10a^3b^7$ by $2a^3b^4$.
- Divide $4m^5x^7v^6 - 12m^4c^2x^8v^4 + 16a^3m^6c^2x^9v^7$ by $-4m^4x^7v^4$.

Find the missing exponent in the quotients.

- $\frac{2x^4}{x^?} = 2x$
- $\frac{2x^3y^2 - x^2y^3}{4xy^?} = \frac{x^2}{2y^2} - \frac{x}{4y}$

Round Up

This leads on to the next few Topics, where you'll divide one polynomial by another polynomial. First, you'll learn how to find the *multiplicative inverse* of a polynomial in Topic 6.3.2.

Polynomials and Negative Powers

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

10.0: Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

What it means for you:

You'll learn how to find the multiplicative inverse of a polynomial, and how to use negative exponents.

Key words:

- polynomial
- monomial
- reciprocal
- exponent

Before you divide one polynomial by another polynomial, you need to know how to write the *multiplicative inverse* of a polynomial.

Finding the Reciprocal of a Polynomial

The **reciprocal** of a polynomial is its **multiplicative inverse**.

To find the **reciprocal of Polynomial A**, you **divide the number 1 by Polynomial A**.

Example 1

Find the reciprocal of each of these polynomials:

a) $2x - 1$ b) $-5x^2 + 3x - 1$

Solution

a) $\frac{1}{2x - 1}$ b) $\frac{1}{-5x^2 + 3x - 1}$

By the **definition of division**, to **divide by a polynomial you need to multiply by the reciprocal** (the **inverse** under multiplication) of that polynomial.

Guided Practice

Find the multiplicative inverse of each of these expressions.

1. ab

2. a^2

3. $2x + 4b$

4. $3x + 1$

5. $8x^3 - 16x^2 + 4$

6. $\frac{1}{-2x^2 + 8x - 9}$

7. $2x^2y^4$

8. $\frac{1}{3a^2b^4c^8}$

9. $\frac{1}{-3x^2y^8c^4}$

You Can Write Reciprocals as Negative Exponents

This rule of exponents gives you another way to express fractions:

$$x^{-a} = \frac{1}{x^a} \text{ provided } x \neq 0$$

Example 2

Simplify the following expressions:

a) 2^{-3}

b) $2x \cdot (3y)^{-1}$

c) $6b^3 \cdot (3ab)^{-2}$

Solution

$$\begin{aligned} \text{a) } 2^{-3} &= \frac{1}{2^3} \\ &= \frac{1}{2 \cdot 2 \cdot 2} \\ &= \frac{1}{8} \end{aligned}$$

$$\begin{aligned} \text{b) } 2x \cdot (3y)^{-1} &= 2x \cdot \frac{1}{3y} \\ &= \frac{2x}{3y} \end{aligned}$$

$$\begin{aligned} \text{c) } 6b^3 \cdot (3ab)^{-2} &= 6b^3 \cdot \frac{1}{(3ab)^2} \\ &= \frac{6b^3}{9a^2b^2} \\ &= \frac{2b}{3a^2} \end{aligned}$$

✓ Guided Practice

Simplify the following expressions.

10. $(2a)^{-2}$

11. $(3ab)^{-3}$

12. $(4x^2y)^{-2}$

13. $(4xy)^{-1}(2x^2)$

14. $(2x^3y)^{-2}(3x)^3$

15. $(16x^5y^7z^{12})(4x^3y^2z^8)^{-1}$

16. $(15a^3b^4x^4)(3a^2b^2x^3)^{-1}$

17. $(40ax^2y^4z^5)^2(4xy^2z^4)^{-2}$

18. $(2a^2b^3c^2d^3)^2(-\frac{1}{3}ab^2cd^2)^{-2}$

19. $(3xyz)^2(2xyz)(6xyz)^{-1}$

20. $(2abc)^3(3a^2bc)^{-2}(a^2b^3c^4)$

21. $(4xy)(8x^2y^3)(x^2y^5)^{-1}$

You Can Write Polynomials as Fractions

A **polynomial** raised to a negative power can also be written as a fraction.

Example 3

Simplify the following expressions:

a) $(a + b) \cdot (b - c)^{-1}$

b) $(m - c)(m + c) \cdot (m - c)^{-1}$

c) $(x - 1)^{-1} \cdot (x + 1)^{-1}$

Solution

$$\text{a) } (a + b) \cdot (b - c)^{-1} = \frac{(a + b)}{(b - c)}$$

$$\text{b) } (m - c)(m + c) \cdot (m - c)^{-1} = \frac{(m - c)(m + c)}{(m - c)} = m + c$$

$$\text{c) } (x - 1)^{-1} \cdot (x + 1)^{-1} = \frac{1}{(x - 1)} \cdot \frac{1}{(x + 1)} = \frac{1}{(x^2 - 1)}$$

Check it out

If a pair of parentheses is raised to a negative power, its entire contents go to the bottom of the fraction.

Check it out

There's more on canceling fractions in 6.3.3.

✓ Guided Practice

Simplify the following expressions.

22. $(x + y) \times (x + y)^{-1}$

23. $(w + v) \times (w - v)^{-1}$

24. $(a - b)^{-1}(a - b)(a + b)$

25. $(a - x)^{-2}(a - x)$

26. $(a - b)(a - b)(a + b)^{-1}(a - b)^{-1}$

27. $(a + x)(a - x)(a - b)^{-1}$

28. $(t - s)^2(t - s)^{-1}(t + s)$

29. $(7y + 6z)(7y - 5z)^{-1}(7y + 6z)^{-2}$

30. $\frac{(8x + 4a)^{-1}(8x - 4a)}{(8x - 4a)(8x - 4a)^{-1}}$

31. $\frac{(4k - j)(4k + j)}{(4k + j)^{-1}(4k - j)}$

✓ Independent Practice

Find the multiplicative inverse of the following.

1. $1.2 + 3.3a$

2. $3x^2 + 5y^3$

3. $-18.6 - 1.5x$

4. $8m^2 - 5mz + 4$

5. $\frac{17}{a - 3.9}$

6. $\frac{4.3x}{8.9y + 4}$

Simplify the following expressions.

7. $15a^2(-3a^2)^{-1}$

8. $(-2x)^2(6ax)^{-2}$

9. $(2y + 3)^{-2} \times (2y + 3)$

10. $4ab \times (3a)^{-1} \times (4b)^{-1}$

11. $(2a^2b^4)^{-1}(4ab)^{-1}(16a^3b^4)$

12. $(a^2b^{-2})^3(a^{-1}b^0)^4(a^{-2}b^{-3})^{-2}$

13. $2x(x - 1)(x - 1)^{-1}(2)^{-1}$

14. $(3z)^{-1}(z - 2)^{-2}(z^2)^3(z - 2)$

15. $(z - 5)^2(3)^{-2}(15)(z - 5)^{-1}$

16. $20(z - 3)^{-2}(z + 3)(z - 3)^3(6)^{-1}$

Round Up

Now you've got all the tools you need. You'll start *dividing one polynomial by another polynomial* using two different methods in Topics 6.3.3 and 6.3.4.

Division by Polynomials — Factoring

California Standards:

10.0: Students add, subtract, multiply, and **divide monomials and polynomials**. Students solve multistep problems, including word problems, by using these techniques.

What it means for you:

You'll divide one polynomial by another polynomial by factoring.

Key words:

- polynomial
- monomial
- factor
- exponent

Now you're ready to *divide a polynomial by another polynomial*. The simplest way to do this is by *factoring the numerator and the denominator*.

Canceling Fractions Helps to Simplify Expressions

If a numerical fraction has a **common factor** in the numerator and

denominator, you can **cancel** it — for example, $\frac{5}{10} = \frac{5 \cdot 1}{5 \cdot 2} = \frac{\cancel{5} \cdot 1}{\cancel{5} \cdot 2} = \frac{1}{2}$.

In the same way, if there are common factors in the numerator and denominator of an algebraic fraction, you can cancel them.

Example 1

Simplify $\frac{(x+1)(2x+3)}{(x+1)}$

Solution

$$\frac{\cancel{(x+1)}(2x+3)}{\cancel{(x+1)}} = 2x + 3$$

This technique's really useful for **dividing polynomials** when the polynomials have already been factored:

Example 2

Divide $(x+4)(1-x)(3x+2)$ by $(1-x)$.

Solution

$$\begin{aligned} & (x+4)(1-x)(3x+2) \div (1-x) \\ &= \frac{(x+4)\cancel{(1-x)}(3x+2)}{\cancel{(1-x)}} \\ &= (x+4)(3x+2) \end{aligned}$$

Check it out:

See Chapter 8 for more about canceling fractions.

✓ Guided Practice

Simplify each expression.

1. $\frac{(x+9)(x-4)}{(x-4)}$

2. $\frac{(2x+1)(2x-1)}{(2x+1)(3x+4)(2x-1)}$

3. $\frac{(5x+2)(x+7)(2x+3)}{(x+7)(2x+3)(5x+2)}$

4. $\frac{(3-2x)(3+2x)}{9-4x^2}$

5. $\frac{y+3}{y+2} \div \frac{y+3}{y+1}$

6. $\frac{x^2-25}{(2x+7)(x+5)(x-5)}$

7. $\frac{(x+5)(x+7)}{(x+1)} \div \frac{(x+7)}{(x-5)}$

8. $\frac{(z+1)(z-1)}{10z^2} \div \frac{(z+1)}{5z^3}$

9. Divide $(x+3)(x+4)$ by $\frac{(x+3)}{8}$.

10. Divide $\frac{3x}{x+2}$ by $\frac{(x+2)^2}{x-4}$.

If It Divides Evenly, the Polynomial Can Be Factored

If you can divide a polynomial evenly, that means there is **no remainder**. This means that it must be possible to **factor** the polynomial (and it means that the divisor is a factor of the polynomial).

Example 3

Given that $(x+1)$ divides evenly into (x^2-4x-5) , find $(x^2-4x-5) \div (x+1)$.

Solution

$$(x^2-4x-5) \div (x+1)$$

$$= \frac{(x^2-4x-5)}{(x+1)}$$

$$= \frac{\cancel{(x+1)}(x-5)}{\cancel{(x+1)}}$$

$$= (x-5)$$

You know that $(x+1)$ is a factor because you're told it divides evenly

Cancel $(x+1)$ from the top and bottom

Check it out:

In Example 3, the numerator, x^2-4x-5 , has been factored, giving $(x+1)(x-5)$. See Section 6.6 for more on factoring quadratics.

✓ Guided Practice

Check it out:

See Sections 6.6 and 6.8 for more on factoring quadratics.

Simplify the quotients by canceling factors.

11. $\frac{4x+8}{x+2}$

12. $\frac{12x^2}{4x+16}$

13. $\frac{20z+4z^2}{z+5}$

14. $\frac{2x^2+12x}{2x}$

15. $\frac{9x}{3x^2-12x}$

16. $\frac{5x-15}{x^2-9}$

17. $\frac{x^2+x-20}{2(x+5)}$

18. $\frac{4x-8}{x^2+2x-8}$

19. $\frac{3a^2-3}{2a^2-2}$

20. Find the ratio of the surface area to the volume of a cube with side length b .

21. Divide $4x - 12$ by $x^2 - 2x - 3$.

✓ Independent Practice

Name the two factors that would divide into each expression below.

1. $x^2 + 7x$

2. $2x - 8$

3. $6a - 15$

4. $6x^2 + 9x$

5. $x^2 + 8x + 15$

6. $a^2 - 81$

Simplify the quotients by canceling factors.

7. $\frac{x-2}{2x-4}$

8. $\frac{2x-5}{6x-15}$

9. $\frac{6a+42}{10a^2+70a}$

10. $\frac{a-8}{a^2-64}$

11. $\frac{4a^2-16}{a+2}$

12. $\frac{x^3+x^2}{x+1}$

13. $\frac{a^2-9}{a^3-3a^2}$

14. $\frac{x^2+9x+8}{x^2+3x+2}$

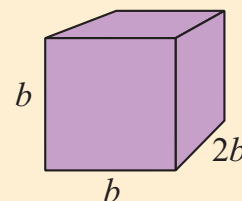
15. $\frac{x^2+7x+12}{x^2+5x+6}$

16. $\frac{x^2-2x-3}{x^2-7x+12}$

17. $\frac{x^2-2x-15}{x^2-25}$

18. $\frac{x^2+x-2}{2x^2-8}$

19. Find the ratio of the surface area to the volume of the rectangular prism shown.



Round Up

This method's most useful for "divides evenly" questions — if a question mentions remainders, the long division method in Topic 6.3.4 is probably better. There are two methods for polynomial division, and you should use the one that makes the most sense for the question you're doing.

✓ Guided Practice

Divide using the long division method.

1. $\frac{2x^2 + 9x - 5}{x + 5}$

2. $\frac{7a^2 + 23a + 6}{a + 3}$

3. $\frac{4x^2 - 8x - 5}{2x - 5}$

4. $\frac{3a^2 + 16a + 16}{a + 4}$

5. $\frac{x^2 - 6x + 8}{x - 4}$

6. $\frac{b^2 + 8b + 15}{b + 3}$

7. $\frac{x^2 - 25}{x + 5}$

8. $\frac{3x^2 - 24x + 48}{x - 4}$

9. Divide $(a^3 - 6a - 4)$ by $(a + 2)$.

10. Divide $(4b^2 + 22b + 12)$ by $(2b + 1)$.

You Can Factor Higher-Degree Polynomials Too

If you've got polynomials of degree higher than 2 (such as cubic equations), it's not always clear how to factor them.

You can use long division to help factor expressions.

For example, if you divide Polynomial A by Polynomial B and get an answer with remainder zero, then Polynomial B is a factor of Polynomial A.

Example 2

Divide $(9m^3 - 3m^2 - 26m - 8)$ by $(3m + 4)$.

Solution

$$\begin{array}{r} 3m^2 - 5m - 2 \\ 3m + 4 \overline{) 9m^3 - 3m^2 - 26m - 8} \\ \underline{-(9m^3 + 12m^2)} \\ -15m^2 - 26m - 8 \\ \underline{-(-15m^2 - 20m)} \\ -6m - 8 \\ \underline{-(-6m - 8)} \\ 0 \end{array}$$

So $(9m^3 - 3m^2 - 26m - 8) \div (3m + 4) = 3m^2 - 5m - 2$

This also means that $(9m^3 - 3m^2 - 26m - 8) = (3m + 4)(3m^2 - 5m - 2)$.
If $(3m^2 - 5m - 2)$ can also be factored, then you can fully factor $(9m^3 - 3m^2 - 26m - 8)$.

Check it out:

Factoring a polynomial fully can be useful for drawing graphs or solving equations.

✓ Guided Practice

Simplify each quotient by dividing using the long division method.

11. $\frac{15x^2 + 17x - 4}{3x + 4}$

12. $\frac{-2x^3 - 3x^2 - x - 6}{x + 2}$

13. $\frac{6y^3 - 19y^2 + 14y - 10}{2y - 5}$

14. $\frac{9x^3 - 3x^2 + 6x + 8}{3x + 2}$

15. Divide $15a^2 + 7ab - 2b^2$ by $5a - b$.

16. Divide $2x^4 - 2x^3 + 3x - 1$ by $2x^3 + 1$.

Find the remaining factors of the polynomial, given that:

17. $(x + 3)$ is a factor of $x^2 - x - 12$.

18. $(x + 4)$ is a factor of $x^3 + 4x^2 - 25x - 100$.

✓ Independent Practice

1. Divide $(x^2 + 16x + 49)$ by $(x + 4)$.

2. Divide $(x^2 + 3x - 6)$ by $(x + 4)$.

Carry out the following divisions using the long division method.

3. $\frac{2x^3 - x^2 - 13x - 6}{x - 3}$

4. $\frac{4a^2 - 2a - 6}{a + 1}$

5. $\frac{6a^3 + 5a^2 + 9}{2a + 3}$

6. $\frac{9x^2 - 3x + 4}{3x + 1}$

7. $\frac{8a^2 + 10a - 1}{2a + 1}$

8. $\frac{-6x^2 - 11x + 9}{3x - 2}$

9. $\frac{x^2 + 2x - 15}{x + 5}$

10. $\frac{3a^2 - 4a - 1}{a + 1}$

11. $\frac{x^2 + 2x + 1}{x + 3}$

12. $\frac{6a^2 - 35a + 36}{3a - 4}$

13. $\frac{10x^2 + 21x + 10}{2x + 3}$

14. $\frac{a^3 - 2a^2 - 33a - 7}{a - 7}$

15. The width of a rectangle is $(x - 3)$ cm and the area is $(3x^2 - 10x + 3)$ cm². What is the length?

16. The base of a triangle is $(x + 2)$ meters and the area is $(x^3 - 6x - 4)$ meters squared. What is the height of the triangle? (Area of a triangle is $A = \frac{1}{2}bh$.)

Find the remaining factors of the polynomial, given that:

17. $(y + 5)$ is a factor of $y^3 + 9y^2 + 23y + 15$.

18. $(y - 2)$ is a factor of $y^3 - 7y^2 + 16y - 12$.

Round Up

*It can sometimes take a while to calculate using **long division** — but sometimes it's the only way of working out a **polynomial division**. If the division looks simple you should try using **factoring** first.*

Division by Polynomials — Applications

California Standards:

10.0: Students add, subtract, multiply, and **divide monomials and polynomials**. Students solve multistep problems, including word problems, by using these techniques.

What it means for you:

You'll use factoring and long division to solve real-life problems involving polynomial division.

Key words:

- polynomial
- monomial
- divisor
- dividend

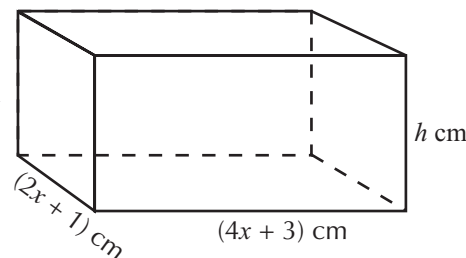
You can use most of the skills you've learned in this Section to solve geometric problems involving polynomials.

Volume Problems Often Involve Polynomials

Example 1

The volume of the box shown is $(40x^3 + 34x^2 - 5x - 6) \text{ cm}^3$.

Write an expression for the height in cm, h , of the box, if it has width $(2x + 1) \text{ cm}$ and length $(4x + 3) \text{ cm}$.



Solution

Start by writing out the formula for volume:

$$\text{volume} = \text{length} \times \text{width} \times \text{height}$$

$$\text{height} = \frac{\text{volume}}{\text{length} \times \text{width}}$$

$$h = \frac{40x^3 + 34x^2 - 5x - 6}{(4x + 3)(2x + 1)}$$

There are a couple of different ways to tackle a division problem like this:

- 1) You could **use long division** to divide the volume **by one of the factors**, leaving you with a **quadratic expression** that you can then **factor by trial and error** — you already know one of the factors.

or

- 2) You could **multiply together** the two factors you know to get a **quadratic expression**. You can then use **long division** to divide the volume **by the quadratic expression**.

In this case, the long division is a bit more complicated than those we tackled on the previous page, but this method does the division in one step rather than two.

Check it out:

You'll cover factoring quadratics later, in Section 6.6.

Check it out:

Which method you choose really just depends on how easy you find long division compared with factoring.

Example 2

Method 1: Use long division to find the height, h , of the box shown in Example 1.

Solution

OK, so you already have the expression $h = \frac{40x^3 + 34x^2 - 5x - 6}{(4x + 3)(2x + 1)}$.

Use long division to divide the volume by one of the factors:

$$\begin{array}{r} 10x^2 + x - 2 \\ 4x + 3 \overline{) 40x^3 + 34x^2 - 5x - 6} \\ \underline{-(40x^3 + 30x^2)} \\ 4x^2 - 5x - 6 \\ \underline{-(4x^2 + 3x)} \\ -8x - 6 \\ \underline{-(-8x - 6)} \\ 0 \end{array}$$

Then you just need to factor the quadratic $(10x^2 + x - 2)$. You know that one of the factors is $(2x + 1)$, from the original question.

By **trial and error**, you can work out that $(10x^2 + x - 2)$ will factor into $(2x + 1)(5x - 2)$.

So the height of the box must be **$(5x - 2)$** cm.

Alternatively, you could do **another long division** to divide $(10x^2 + x - 2)$ by $(2x + 1)$.

Example 3

Method 2: Use the multiplying factors method to find the height, h , of the box shown in Example 1.

Solution

Start by **multiplying together the factors** you know:

$$\begin{aligned} (4x + 3)(2x + 1) &= 8x^2 + 4x + 6x + 3 \\ &= 8x^2 + 10x + 3 \end{aligned}$$

Then use **long division** to divide the volume by the product of the two factors:

$$\begin{array}{r} 5x - 2 \\ 8x^2 + 10x + 3 \overline{) 40x^3 + 34x^2 - 5x - 6} \\ \underline{-(40x^3 + 50x^2 + 15x)} \\ -16x^2 - 20x - 6 \\ \underline{-(-16x^2 - 20x - 6)} \\ 0 \end{array}$$

So the answer is **$h = (5x - 2)$** cm.

Check it out:

Again, you start with the leading term of each polynomial.

✓ Guided Practice

1. If the area of a rectangle is $(6y^2 + 29y + 35)$ square units, and its length is $(3y + 7)$ units, find the width of the rectangle.
2. The volume of a rectangular box is $(60x^3 + 203x^2 + 191x + 36)$ cubic inches. Find the height of the box if the area B of the base is $(15x^2 + 47x + 36)$ square inches. [Hint: $V = Bh$]
3. The area of a circle is $(\pi x^2 - 10\pi x + 25\pi)$ m². Show that the radius is $(x - 5)$ m.
4. The volume of a prism is $(x^3 + 8x^2 + 19x + 12)$ m³. Find the area of the base if the height is $(x + 4)$ m.
5. A rectangular prism has volume $(2b^3 - 7b^2 + 2b + 3)$ m³, height $(b - 1)$ m, and length $(2b + 1)$ m. Find its width.

✓ Independent Practice

1. The width of a rectangle is $(3x + 4)$ m. If the area of the rectangle is $(6x^2 + 5x - 4)$ m², what is the length?
2. The area, A , of a triangle is given by the formula $2A = bh$, where b is the base length and h is the height. Find the length of the base if $A = (3x^2 - 16x + 16)$ ft² and $h = (2x - 8)$ ft.
3. Find the height of a rectangle with area $(2x^2 - 9x + 9)$ ft² and width $(2x - 3)$ ft.
4. The volume of a prism is $(2x^3 + x^2 - 3x)$ m³. If the area of the base is $(x^2 - x)$ m², what is the height of the prism?
5. A rectangular prism has volume $(b^3 + 9b^2 + 26b + 24)$ m³, width $(b + 2)$ m, and length $(b + 4)$ m. Find its height.
6. The volume of a prism is $(144s^3 + 108s^2 - 4s - 3)$ m³. If the area of the base is $(36s^2 - 1)$ m², what is the height?
7. The volume of a cylinder is $(2\pi p^3 + 7\pi p^2 + 8\pi p + 3\pi)$ in³. Find the area of the base if the height is $(2p + 3)$ in.
8. The volume of a cylinder is $(18\pi y^3 - 3\pi y^2 - 28\pi y - 12\pi)$ ft³. Find the height of the cylinder if the area of the base is $(9\pi y^2 + 12\pi y + 4\pi)$ ft².

Round Up

In this Section you've seen *two* good ways of *dividing one polynomial by another polynomial* — *factoring and long division*.

Topic 6.4.1

Section 6.4 Special Products of Two Binomials

California Standards:

2.0: Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

10.0: Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.

11.0 Students apply basic factoring techniques to second and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

What it means for you:

You'll learn how to use special cases of binomial multiplication to save time in calculations.

Key words:

- binomial
- difference of two squares

Check it out:

$(a + b)^2$ expands to give a perfect square trinomial — see Topic 6.8.2.

This Topic is all about special cases of binomial multiplication. Knowing how to expand these special products will save you time when you're dealing with binomials later in Algebra I.

Remember These Three Special Binomial Products

Given any real numbers a and b , then:

$$(a + b)^2 = a^2 + 2ab + b^2$$

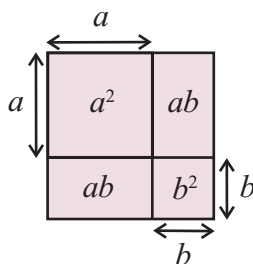
$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

When You Expand $(a + b)^2$, You Always Get an ab -Term

$$\begin{aligned} (a + b)^2 &= (a + b)(a + b) \\ &= a^2 + ab + ba + b^2 \quad \text{Using the distributive property} \\ &= a^2 + 2ab + b^2 \end{aligned}$$

You can relate this equation to the area of a square:



$(a + b)^2$ is the same as the area of this large square — add the areas of the two smaller squares, a^2 and b^2 , and the two rectangles, $2 \times ab$.

Example 1

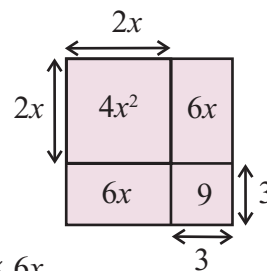
Expand and simplify $(2x + 3)^2$.

Solution

Put the expression in the form $(a + b)(a + b)$:

$$\begin{aligned} (2x + 3)^2 &= (2x + 3)(2x + 3) \\ &= 4x^2 + 6x + 6x + 9 \\ &= 4x^2 + 12x + 9 \end{aligned}$$

$(2x + 3)^2$ is the same as the area of the large square — add the areas of the two smaller squares, $4x^2$ and 9 , and the two rectangles, $2 \times 6x$.



✓ Guided Practice

Find and simplify each product.

1. $(x + 5)^2$

2. $(2y + 3)^2$

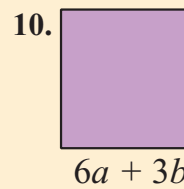
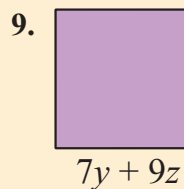
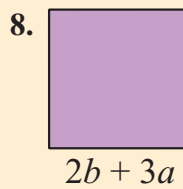
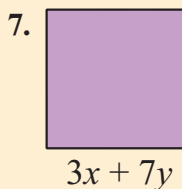
3. $(3y + 1)^2$

4. $(9y + z)^2$

5. $(n^2 + 4n)^2$

6. $(3t + 6b)^2$

Find the areas of the squares below.



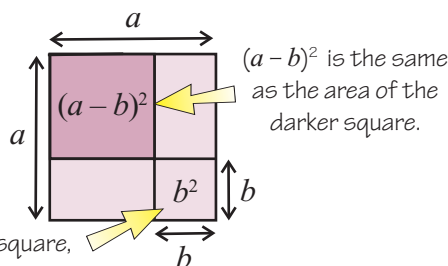
When You Expand $(a - b)^2$, the ab -Term is Negative

Check it out:

$(a - b)^2$ also expands to give a perfect square trinomial — see Topic 6.8.2.

$$\begin{aligned} (a - b)^2 &= (a - b)(a - b) \\ &= a^2 - ab - ba + b^2 && \text{Using the distributive property} \\ &= a^2 - 2ab + b^2 \end{aligned}$$

You can also relate this equation to the area of a smaller square:



To find the area of the darker square, you can subtract two rectangles of area ab — but then you have to add back on an area of b^2 (this small square).

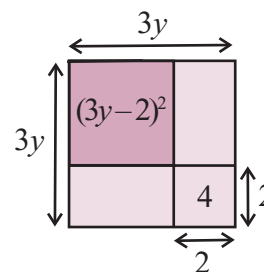
Example 2

Expand $(3y - 2)^2$.

Solution

Put the expression in the form $(a - b)(a - b)$:

$$\begin{aligned} (3y - 2)^2 &= (3y - 2)(3y - 2) \\ &= (3y)^2 - 2(3y \times 2) + 2^2 \\ &= 9y^2 - 2(6y) + 2^2 \\ &= 9y^2 - 12y + 4 \end{aligned}$$



In the diagram on the right, $(3y - 2)^2$ is the same as the area of the darker square.

✓ Guided Practice

Find and simplify each product.

11. $(m - 4)^2$

12. $(y - 2)^2$

13. $(4k - 5)^2$

14. $(2a - 5)^2$

15. $(7d - f)^2$

16. $(xy - 3)^2$

17. $(x^2 - y)^2$

18. $(9x - 3y)^2$

19. $(3x^2 - 4)^2$

20. Find the area of the square shown.



$(3x - 1)$ in.

Check it out:

This is called a difference of two squares, and it can also be used in factoring expressions — see Topic 6.8.1.

There's No ab -Term When You Expand $(a + b)(a - b)$

$$\begin{aligned}(a + b)(a - b) &= a^2 - ab + ba - b^2 && \text{Using the distributive property} \\ &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

The fact that **no ab -term** is left at all makes it unusual, but also very useful if you remember it.

Example 3

Multiply out $(4m + 3)(4m - 3)$.

Solution

The expression is already in the form $(a + b)(a - b)$, so you can convert it to the form $a^2 - b^2$:

$$\begin{aligned}(4m + 3)(4m - 3) &= (4m)^2 - 3^2 \\ &= 16m^2 - 9\end{aligned}$$

✓ Guided Practice

Find and simplify each product.

21. $(m - v)(m + v)$

22. $(x + 5)(x - 5)$

23. $(3y + x)(3y - x)$

24. $(k - 6t)(k + 6t)$

25. $(3x - 9y)(3x + 9y)$

26. $(6x + 6y)(6x - 6y)$

27. $(x + 3x^2)(x - 3x^2)$

28. $(9p^2 - 2)(9p^2 + 2)$

29. $(x^2 - x)(x^2 + x)$

30. $(7a^2 + b)(7a^2 - b)$

You Can Use These Standard Equations as Shortcuts

The good thing about knowing these standard equations is that **you don't need to do all the work each time** — you can save time by using the three special products.

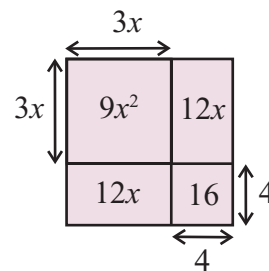
Example 4

Find the area of a square which has side lengths of $(3x + 4)$ inches.

Solution

Using $(a + b)^2 = a^2 + 2ab + b^2$ and putting in $(3x + 4)$ instead of $(a + b)$ you get:

$$\begin{aligned} \text{Area of square} &= (3x + 4)^2 \\ &= 9x^2 + (2 \times 12x) + 16 \\ &= \mathbf{(9x^2 + 24x + 16) \text{ in}^2} \end{aligned}$$



Check it out:

In this case, $a^2 = (3x)^2$,
 $2ab = 2 \times (3x \times 4)$, and $b^2 = 4^2$.

Example 5

Multiply $(4x - 7)$ by $(4x - 7)$.

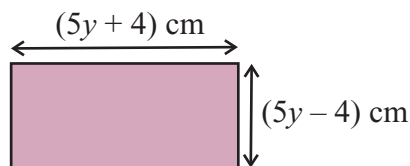
Solution

Using $(a - b)^2 = a^2 - 2ab + b^2$ and putting in $(4x - 7)$ instead of $(a - b)$ you get:

$$\begin{aligned} &(4x - 7)^2 \\ &= 16x^2 - (2 \times 28x) + 49 \\ &= \mathbf{16x^2 - 56x + 49} \end{aligned}$$

Example 6

Find the area of this rectangle:



Solution

Using $(a + b)(a - b) = a^2 - b^2$ and putting in $5y$ for a and 4 for b you get:

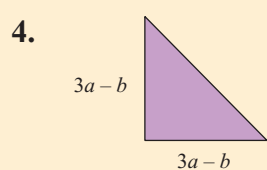
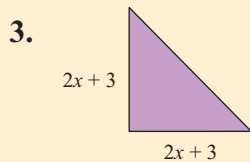
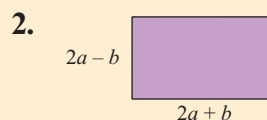
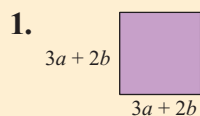
$$\begin{aligned} \text{Area} &= (5y + 4)(5y - 4) \\ &= (5y)^2 - 4^2 \\ &= \mathbf{(25y^2 - 16) \text{ cm}^2} \end{aligned}$$

Check it out:

Using the equation here is a little shorter than expanding the parentheses.

Independent Practice

Find the areas of these shapes.



Find and simplify each product:

5. $(2r - 3)(2r + 3)$

6. $(x^3 + 2)(x^3 - 2)$

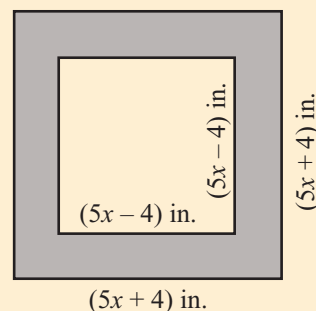
7. $(6x^2 - 1)^2$

8. $(z^2 - z)^2$

9. $(x^2 - 3x)(x^2 + 3x)$

10. $(2x^2 - 3x)(2x^2 + 3x)$

11. Find the area of this shaded region:



The area of a circle with radius r is given by the formula $A = \pi r^2$. Find the areas of these circles, giving your answers in terms of π :

12. A circle with radius $(3x + 4)$.

13. A circle with radius $(2x - 7)$.

14. A circle with radius $(2a + b)$.

15. Find the coefficient of ab in the product $(5a - 4b)^2$.

16. Find the coefficient of mc in the product $(4m - 3c)(4m + 3c)$.

Find and simplify each product:

17. $(3x + 7)^2 - (3x - 7)^2$

18. $(2x^a + 1)^2$

19. $(k^m - x^m)^2$

20. Nicole has a circle of card with a radius of $(4x + 5)$ cm.

She makes the circle into a ring by cutting a circular hole in the middle with a radius of $(4x - 5)$ cm. Find the area of the ring, leaving your answer in terms of π .

Round Up

The main reason for learning these special products is to make your life easier when you multiply two binomials or factor quadratics. You'll come across them throughout the rest of Algebra I.

Topic 6.5.1

Section 6.5 Factors of Monomials

California Standards:

11.0 Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

What it means for you:

You'll learn how to find factors of a monomial.

Key words:

- factor
- monomial
- greatest common factor
- prime factor

Don't forget:

A monomial is a single term expression made up of a number or a product of a number and variables — such as 13 , $2x^2$ or $-x^3yn^4$.

Check it out:

Factors that can't be factored themselves, like 2 , 3 , 5 , x and y , are called prime factors. It's these prime factors you need to use here.

In previous Topics you've already done lots of manipulation of polynomials — but you can often make manipulations easier by breaking down polynomials into smaller chunks.

In this Topic you'll break down monomials by factoring.

Numbers Have Factors — and So Do Monomials

Sometimes a number can be written as the **product of two or more smaller numbers**. Those smaller numbers are called **factors** of that number.

For example, $6 = 2 \times 3$ — so **2** and **3** are **factors of 6**.

The same is true for **monomials**. Unless they're prime numbers, monomials can be written as the **product of two or more numbers or letters**.

Those smaller numbers or letters are called **factors** of that monomial.

For example, $3xy = 3 \times x \times y$ — so **3**, **x**, and **y** are **factors of 3xy**.

✓ Guided Practice

Write down all of the factors of each of these numbers:

1. 8

2. 10

3. 15

4. 16

5. 11

6. 24

Write down each monomial as a product of the smallest possible factors:

7. $3x$

8. $7z$

9. $6p$

10. $5xy$

11. $12uv$

12. $20mn$

The GCF is a Divisor of Each Monomial

The **greatest common factor (GCF)** of a set of monomials is the **largest possible divisor** of all monomials in the set.

Example 1

Find the greatest common factor of $12x^2y^2$, $18x^3y^2$, and $30x^4y^4$.

Solution

Start by writing down each monomial as a product of the smallest factors possible:

$$\begin{aligned} 12x^2y^2 &= 2 \times 2 \times 3 \times x \times x \times y \times y \\ 18x^3y^2 &= 2 \times 3 \times 3 \times x \times x \times x \times y \times y \\ 30x^4y^4 &= 2 \times 3 \times 5 \times x \times x \times x \times x \times y \times y \times y \times y \end{aligned}$$

Then list all the numbers that are factors of all three terms: 2 , 3 , x , x , y , y .

These are called the **common factors**.

Example 1 continued

The greatest common factor is the product of all the common factors:

$$\text{GCF} = 2 \times 3 \times x \times x \times y \times y = 6x^2y^2$$

In other words, $6x^2y^2$ is the largest possible divisor of $12x^2y^2$, $18x^3y^2$, and $30x^4y^4$.

✓ Guided Practice

Use the method from Example 1 to write down the greatest common factor of each set of products:

13. 12, 24, 42

14. $9ab^2$, $15a^2b^2$, $12ab$

15. $6m^2cv^2$, $10m^2c^2v$, $4m^2c^2v^3$

16. $5mx^2t$, $15m^2xt^2$, $20mxt$

17. $9x^3y^2$, $27x^2y^3$

Another Way to Find the GCF of Two or More Monomials

You can also find the GCF of two or more monomials by simply **multiplying** together the **GCFs of each of the different parts**.

Example 2

Find the greatest common factor of $12x^2y^2$, $18x^3y^2$, and $30x^4y^4$.

Solution

The GCF of 12, 18, and 30 is **6**.

The GCF of x^2 , x^3 , and x^4 is **x^2** .

The GCF of y^2 and y^4 is **y^2** .

So, the GCF of $12x^2y^2$, $18x^3y^2$, and $30x^4y^4$ is $6 \times x^2 \times y^2 = 6x^2y^2$.

✓ Guided Practice

Use the method from Example 2 to write down the greatest common factor of each set of products:

18. b^3m^2cv , bm^2v

19. $2(m+1)$, $-3(m+1)$, $(m+1)^2$

20. $8(v-1)^2$, $4(v-1)^3$, $12(v-1)^2$

21. $6x^2yz$, $15xz$

22. $21x^4y^4z^4$, $42x^3y^4z^5$, $14x^6y^3z^2$

Check it out:

You could also calculate all of these GCFs using the method in Example 1.

✓ Independent Practice

Write down all of the factors of each of these numbers:

- | | | |
|-------|-------|-------|
| 1. 25 | 2. 12 | 3. 36 |
| 4. 67 | 5. 80 | 6. 70 |

Write each of these as a product of prime factors:

- | | | |
|--------|---------|---------|
| 7. 48 | 8. 72 | 9. 120 |
| 10. 66 | 11. 450 | 12. 800 |

Write each monomial as a product of the smallest possible factors:

- | | | |
|---------------|-----------------|-------------------|
| 13. $66z^2$ | 14. $4b^3d^2$ | 15. $-102x^3y$ |
| 16. $-98a^2b$ | 17. $64y^3z^3$ | 18. $-80rs^5$ |
| 19. $3x^2yz$ | 20. $16pq^3r^3$ | 21. $-100f^2gh^4$ |

Check it out:

You can use either method of finding the greatest common factor here — it's up to you.

Write down the greatest common factor of each set of products:

- | | |
|--|---------------------------------|
| 22. 18, 36 | 23. 84, 75 |
| 24. 95, 304 | 25. $17a, 34a^2$ |
| 26. $21p^2q, 35pq^2$ | 27. $12an^2, 40a^4$ |
| 28. $-60r^2s^2t^2, 45r^3t^3$ | 29. 18, 30, 54 |
| 30. $14a^2b^3, 20a^3b^2c^2, 35ab^3c^2$ | 31. $18x^2, 30x^3y^2, 54y^3$ |
| 32. $14a^2b^2, 18ab, 2a^3b^3$ | 33. $32m^2n^3, 8m^2n, 56m^3n^2$ |

34. The area of a rectangle is 116 square inches.

What are its possible whole number dimensions?

35. The area of a rectangle is 1363 square centimeters.

If the measures of the length and width are both prime numbers, what are the dimensions of the rectangle?

36. Marisela is planning to have 100 tomato plants in her garden.

In what ways can she arrange them in rows so that she has the same number of plants in each row, at least 5 rows of plants, and at least 5 plants in each row?

37. A walkway is being paved using 2-ft-by-2-ft paving stones.

If the length of the walkway is 70 ft longer than the width and its area is 6000 ft^2 , how many paving stones make up the length and the width of the walkway?

Round Up

Factoring is the best way of working out which smaller parts make up a number or monomial. In the next few Topics you'll use factoring to break down full expressions, which makes it much easier to do tricky jobs like solving some kinds of equations.

Simple Factors of Polynomials

California Standards:

11.0 Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

What it means for you:

You'll learn how to find factors of a polynomial.

Key words:

- factor
- monomial
- polynomial
- greatest common factor

The Topics in this Section are all about making your work a lot easier.

Factoring polynomials can make them simpler, which means they're easier to manipulate.

Polynomials Can (Sometimes) Be Factored Too

When a **polynomial** can be expressed as the **product of two or more numbers, monomials, or polynomials**, those smaller numbers, monomials, or polynomials are called **factors** of that polynomial.

For example, if Polynomial A can be written as either:

$$2x^2 + x \quad \text{or} \quad x(2x + 1),$$

this means that **x is a factor of Polynomial A.**

It also means that **$(2x + 1)$ is a factor of Polynomial A.**

To find the factors of a polynomial, start by finding the greatest common factor of all the terms.

Example 1

Factor $6m^3 - 4m^2$.

Solution

$$\begin{aligned} 6m^3 &= 2 \times 3 \times m \times m \times m \\ 4m^2 &= 2 \times 2 \times m \times m \end{aligned}$$

The factors that are present in both terms are 2, m , and m — so 2, m , and m are factors of both terms.

So the GCF = $2 \times m \times m = 2m^2$

Next you need to rewrite the expression with the factor taken out:

$$\begin{aligned} 6m^3 &= 2 \times 3 \times m \times m \times m = (2 \times m \times m) \times 3 \times m = 2m^2 \times 3m \\ 4m^2 &= 2 \times 2 \times m \times m = (2 \times m \times m) \times 2 = 2m^2 \times 2 \end{aligned}$$

$$\begin{aligned} \text{So } 6m^3 - 4m^2 &= (2m^2 \times 3m) - (2m^2 \times 2) \\ &= 2m^2(3m - 2) \end{aligned}$$

It's always worth checking your factoring by multiplying out again:

$$\begin{aligned} 2m^2(3m - 2) &= (2 \times m^2 \times 3 \times m) - (2 \times m^2 \times 2) \\ &= (6 \times m^{2+1}) - (4 \times m^2) = 6m^3 - 4m^2 \end{aligned}$$

✓ Guided Practice

In each polynomial, find the greatest common factor of the terms.

1. $12y^2 - 3y$ 2. $a^3 + 3a$ 3. $14a^3 - 28a^2 + 56a$

4. $16y^2 - 24y^3$ 5. $60x^3 + 24x^2 + 16x$ 6. $18y^5 + 6y^4 + 3y^2$

Use your answers from Exercises 1 – 6 to factor the following:

7. $12y^2 - 3y$ 8. $a^3 + 3a$ 9. $14a^3 - 28a^2 + 56a$

10. $16y^2 - 24y^3$ 11. $60x^3 + 24x^2 + 16x$ 12. $18y^5 + 6y^4 + 3y^2$

An Alternative Method for Taking Out Factors

Again, find the GCF by **multiplying** together the **GCFs of each of the different parts** — the coefficients and the variables. Then just rewrite the expression with the GCF taken out, as before.

Check it out:

Example 2 covers the same problem as Example 1, but this time using the GCF of each of the different parts.

Example 2

Factor $6m^3 - 4m^2$.

Solution

The greatest common factor of 6 and 4 is 2.

Since both terms contain m , the lowest power will be a factor.

So $\text{GCF} = 2 \times m \times m = 2m^2$

To write $6m^3 - 4m^2$ as a factored expression, multiply and divide by $2m^2$, as shown:

$$\begin{aligned} 6m^3 - 4m^2 &= 2m^2 \left(\frac{6m^3}{2m^2} - \frac{4m^2}{2m^2} \right) \\ &= 2m^2(3m - 2) \end{aligned}$$

✓ Guided Practice

Factor each polynomial below.

13. $x^2 - 4x$

14. $x^2 - x$

15. $24x^3 - 15x^2 + 6x$

16. $8x^3 + 2x^2 + 4x$

17. $4a^3 - 6a^2 + 6a$

18. $14b^2 + 7b - 21$

19. $6b^3 - 3b^2 + 12b$

20. $a^4 + a^5 + 5a^3$

You Can Take Out Polynomial Factors Too

At the start of the Topic you saw that $2x^2 + x$ can be written as $x(2x + 1)$.

This means that both x (a monomial) and $2x + 1$ (a polynomial) are **factors** of $2x^2 + x$. The next few examples are about finding polynomial factors of the form $(ax + b)$.

Example 3

Factor $(d - 1)x^2 + (d - 1)x + (d - 1)$.

Solution

Each term is a product of $(d - 1)$ and something else, so $(d - 1)$ is a common factor.

To write $(d - 1)x^2 + (d - 1)x + (d - 1)$ as a factored expression, put the $(d - 1)$ outside parentheses and divide everything inside the parentheses by $(d - 1)$, as shown:

$$\begin{aligned} & (d - 1)x^2 + (d - 1)x + (d - 1) \\ &= (d - 1) \left(\frac{\cancel{(d - 1)}x^2}{\cancel{(d - 1)}} + \frac{\cancel{(d - 1)}x}{\cancel{(d - 1)}} + \frac{\cancel{(d - 1)}}{\cancel{(d - 1)}} \right) \\ &= (d - 1)(x^2 + x + 1) \end{aligned}$$

Example 4

Show that $(a + b)$ is a factor of $ac + bc + ad + bd$.

Solution

You've been given the factor, so try writing the polynomial as a factored expression. If you can do that, you'll have shown that $(a + b)$ is a factor.

Take the $(a + b)$ outside parentheses, as above:

$$\begin{aligned} & ac + bc + ad + bd \\ &= (a + b) \left(\frac{ac}{(a + b)} + \frac{bc}{(a + b)} + \frac{ad}{(a + b)} + \frac{bd}{(a + b)} \right) \\ &= (a + b) \left(\frac{ac + bc}{(a + b)} + \frac{ad + bd}{(a + b)} \right) \\ &= (a + b) \left(\frac{\cancel{(a + b)}c}{\cancel{(a + b)}} + \frac{\cancel{(a + b)}d}{\cancel{(a + b)}} \right) \\ &= (a + b)(c + d) \end{aligned}$$

Therefore $(a + b)$ is a factor.

Check it out:

You're aiming to cancel $(a + b)$ from the top.

Check it out:

Note that grouping the fractions differently helps to show $(a + b)$ as a factor.

Example 5

Factor and simplify the following expression:

$$(x - 2)(x + 2)x + (x - 2)x + (x - 2)(x + 2).$$

Solution

Each term is a product of $(x - 2)$ and something else, so $(x - 2)$ is a common factor.

To write $(x - 2)(x + 2)x + (x - 2)x + (x - 2)(x + 2)$ as a factored expression, write the $(x - 2)$ outside parentheses, then divide all terms by $(x - 2)$:

$$\begin{aligned} & (x - 2)(x + 2)x + (x - 2)x + (x - 2)(x + 2) \\ &= (x - 2) \left(\frac{\cancel{(x - 2)}(x + 2)x}{\cancel{(x - 2)}} + \frac{\cancel{(x - 2)}x}{\cancel{(x - 2)}} + \frac{\cancel{(x - 2)}(x + 2)}{\cancel{(x - 2)}} \right) \\ &= (x - 2)[(x + 2)x + x + (x + 2)] \\ &= (x - 2)(x^2 + 2x + x + x + 2) \\ &= \mathbf{(x - 2)(x^2 + 4x + 2)} \end{aligned}$$

✓ Guided Practice

Factor and simplify.

21. $x(2x + 1) + 3(2x + 1)$

22. $3y^2(2 - 3x) + y(2 - 3x) + 5(2 - 3x)$

23. $2x^4(5x - 3) - x^2(5x - 3) + (5x - 3)$

24. $2a(3a - 1) + 6(3a - 1)$

25. $(4 - x)x^2 + (4 - x)2x + (4 - x)1$

26. Show that $(x + 3)$ is a factor of $x^2 + 2x + 3x + 6$.

27. Show that $(y + 2)$ is a factor of $y^2 + y + 2y + 2$.

28. Show that $(3x - 4)$ is a factor of $6x^2 + 9x - 8x - 12$.

Factor and simplify the following expressions.

29. $(x + 3)x + (x + 3)(x - 1) + (x + 3)x - 2(x + 3)$

30. $(2x - 1)2x + (2x - 1)(2x - 1) + (2x - 1)x - (2x - 1)$

31. $(x^3 + x^2)(x + 1) - (x^3 + x^2)(x^2 + x) + (x^3 + x^2)(x^2 + 2x + 3)$

Independent Practice

In each polynomial, find the greatest common factor of the terms.

1. $8x^2 - 12yx$ 2. $81x^3 + 54x^2$ 3. $21x^3yz - 35x^2y + 70xyz$

Factor each polynomial below.

4. $2x - 6$

5. $6x^2 - 12x$

6. $5(c + 1) - 2y(c + 1)$

7. $6y^2 - 12y^3 + 18y$

8. $x^2(k + 3) + (k + 3)$

9. $k(y - 3) - m(y - 3)$

10. $(x + 1)^2 - 2(x + 1)^3$

11. $m(y - 5)^2 - (y - 5)$

12. $(x^2 - 2x) + (4x - 8)$

13. $(y^2 + 3y) + (3y + 9)$

14. $(2my - 3mx) + (-4y + 6x)$

15. $-2m(x + 1) + k(x + 1)$

16. $x^5 + 3x^3 + 2x^4$

17. $8y^2x + 4yx^2 + 4y^2x^2$

18. $3x^3 - 6x^2 + 9x$

19. $4x^5 - 4x^3 + 16x^2$

20. $2m^3n - 6m^2n^2 + 10mn$

21. Show that $(x + 4)$ is a factor of $x^2 - 5x + 4x - 20$.

22. Show that $(2x + 5)$ is a factor of $2x^2 - 2x + 5x - 5$.

23. Show that $(3x - 1)$ is a factor of $6x^2 + 3x - 2x - 1$.

24. Show that $(a - b)$ is a factor of $2ac - ad - 2bc + bd$.

Factor and simplify the following expressions.

25. $(4x^2 + 3)(2x + 1) + (4x^2 + 3)(2x + 2) + 8(4x^2 + 3)$

26. $20(4a + 3b) - (x + 1)(4a + 3b)$
 $+ (x^2 + x + 8)(4a + 3b) + (4a + 3b)$

Round Up

The *greatest common factor* is really useful when you're trying to factor polynomials, because it's always the *best factor* to use. In the next Section you'll see that you can also factor more complicated polynomials like *quadratics*.

Topic 6.6.1

Section 6.6 Factoring Quadratics

California Standards:

11.0: Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

What it means for you:

You'll learn how to factor simple quadratic expressions.

Key words:

- quadratic
- polynomial
- factor
- binomial

Check it out:

The order of the parentheses doesn't matter — you could have $(x + 3)$ first.

In Section 6.5 you worked out *common factors* of polynomials.

Factoring quadratics follows the same rules, but you have to watch out for the *squared terms*.

Polynomials as Products of Two or More Factors

A **quadratic polynomial** has **degree two**, such as $2x^2 - x + 7$ or $x^2 + 12$. Some quadratics can be factored — in other words they can be expressed as a product of two linear factors.

Suppose $x^2 + bx + c$ can be written in the form $(x + m)(x + n)$.

Then:

$$\begin{aligned}x^2 + bx + c &= (x + m)(x + n) && \leftarrow b, c, m, \text{ and } n \text{ are numbers} \\ &= x(x + n) + m(x + n) && \leftarrow \text{Expand out the parentheses using} \\ &= x^2 + nx + mx + mn && \leftarrow \text{the distributive property}\end{aligned}$$

So, $x^2 + bx + c = x^2 + (m + n)x + mn$ \leftarrow Therefore $b = m + n$ and $c = mn$

So, to factor $x^2 + bx + c$, you need to find two numbers, m and n , that multiply together to give c , and that also add together to give b .

Example 1

Factor $x^2 + 5x + 6$.

Solution

The expression is $x^2 + 5x + 6$, so find two numbers that **add up to 5** and that also **multiply to give 6**.

The numbers 2 and 3 multiply together to give 6 and add together to give 5. You can now factor the quadratic, using these two numbers:

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

To check whether the binomial factors are correct, multiply out the parentheses and then simplify the product:

$$\begin{aligned}(x + 2)(x + 3) &= x(x + 3) + 2(x + 3) && \text{Using the distributive property} \\ &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6\end{aligned}$$

This is the same as the original expression, so the factors are correct.

Example 2Factor $x^2 - x - 6$.**Solution**

Find two numbers that multiply to give -6 and add to give -1 , the coefficient of x . Because c is negative (-6), one number must be positive and the other negative.

$$x^2 - x - 6 = (x - 3)(x + 2)$$

Check whether the binomial factors are correct:

$$\begin{aligned} (x - 3)(x + 2) &= x(x + 2) - 3(x + 2) && \text{Using the distributive property} \\ &= x^2 + 2x - 3x - 6 \\ &= x^2 - x - 6 \end{aligned}$$

This is the same as the original expression, so the factors are correct.

Check it out:

The numbers -3 and $+2$ multiply together to give -6 and add together to give -1 .

Example 3Factor $x^2 - 5x + 6$.**Solution**

Find two numbers that multiply to give $+6$ and add to give -5 , the coefficient of x . Because c is positive (6) but b is negative, the numbers must both be negative.

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

Check whether the binomial factors are correct:

$$\begin{aligned} (x - 2)(x - 3) &= x(x - 3) - 2(x - 3) && \text{Using the distributive property} \\ &= x^2 - 3x - 2x + 6 \\ &= x^2 - 5x + 6 \end{aligned}$$

This is the same as the original expression, so the factors are correct.

Check it out:

The numbers -2 and -3 multiply together to give $+6$ and add together to give -5 .

Example 4Factor $x^2 + 2x - 8$.**Solution**

Find two numbers that multiply to give -8 and add to give $+2$, the coefficient of x .

$$x^2 + 2x - 8 = (x + 4)(x - 2)$$

Check whether the binomial factors are correct:

$$\begin{aligned} (x + 4)(x - 2) &= x(x - 2) + 4(x - 2) && \text{Using the distributive property} \\ &= x^2 - 2x + 4x - 8 \\ &= x^2 + 2x - 8 \end{aligned}$$

This is the same as the original expression, so the factors are correct.

Check it out:

The numbers $+4$ and -2 multiply together to give -8 and add together to give $+2$.

✓ Guided Practice

Factor each expression below.

- | | |
|---------------------|---------------------|
| 1. $a^2 + 7a + 10$ | 2. $x^2 + 7x + 12$ |
| 3. $x^2 - 17x + 72$ | 4. $x^2 + x - 42$ |
| 5. $b^2 + 2b - 24$ | 6. $a^2 - a - 42$ |
| 7. $x^2 - 15x + 54$ | 8. $m^2 + 2m - 63$ |
| 9. $t^2 + 16t + 55$ | 10. $p^2 + 9p - 10$ |
| 11. $x^2 - 3x - 18$ | 12. $p^2 + p - 56$ |
| 13. $x^2 - 2x - 15$ | 14. $n^2 - 5n + 4$ |
| 15. $x^2 - 4$ | 16. $x^2 - 25$ |
| 17. $4x^2 - 64$ | 18. $9a^2 - 36$ |
| 19. $x^2 - 49a^2$ | 20. $4a^2 - 100b^2$ |

✓ Independent Practice

Find the value of ? in the problems below.

- | | |
|--------------------------------------|--------------------------------------|
| 1. $x^2 + 3x - 4 = (x + 4)(x + ?)$ | 2. $a^2 - 2a - 8 = (a + ?)(a + 2)$ |
| 3. $x^2 + 16x - 17 = (x + ?)(x - 1)$ | 4. $x^2 - 14x - 32 = (x + 2)(x + ?)$ |
| 5. $a^2 + 6a - 40 = (a - 4)(a + ?)$ | |

Factor each expression below.

- | | | |
|----------------------|-----------------------|----------------------|
| 6. $x^2 - 121$ | 7. $4c^2 - 64$ | 8. $16a^2 - 225$ |
| 9. $x^2 + 2x + 1$ | 10. $x^2 + 8x + 16$ | 11. $b^2 - 10b + 25$ |
| 12. $d^2 + 21d + 38$ | 13. $x^2 - 13x + 42$ | 14. $a^2 - 18a + 45$ |
| 15. $a^2 - 16a + 48$ | 16. $x^2 + 18x + 17$ | 17. $x^2 - 24x + 80$ |
| 18. $a^2 + 5a - 24$ | 19. $b^2 - 19b - 120$ | 20. $x^2 + 14x - 72$ |

21. Determine whether $(x + 3)$ is a factor of $x^2 - 2x - 15$.
22. If $2n^3 - 5$ is a factor of $12n^5 + 2n^4 - 30n^2 - 5n$, find the other factors.
23. If $(8n - 3)$ is a factor of $8n^3 - 3n^2 - 8n + 3$, find the other factors.
24. If $(2x + 5)$ is a factor of $2x^3 + 15x^2 + 13x - 30$, find the other factors.
25. If $(a - 1)$ is a factor of $a^3 - 6a^2 + 9a - 4$, find the other factors.
26. If $(x - 2)$ is factor of $x^3 + 5x^2 - 32x + 36$, find the other factors.

What can you say about the signs of a and b when:

27. $x^2 + 9x + 16 = (x + a)(x + b)$,
28. $x^2 - 4x + 6 = (x + a)(x + b)$,
29. $x^2 + 10x - 75 = (x + a)(x + b)$,
30. $x^2 - 4x - 32 = (x + a)(x + b)$?

Round Up

The method in this Topic only works for *quadratic expressions* that have an x^2 term with a coefficient of 1 (so it's usually written just as x^2 rather than $1x^2$). In the next Topic you'll see how to deal with other types of quadratics.

Factoring Quadratics

— $ax^2 + bx + c$

California Standards:

11.0: Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

What it means for you:

You'll learn how to factor more complicated quadratic expressions.

Key words:

- quadratic
- polynomial
- factor
- trial and error

The method in Topic 6.6.1 for factoring a quadratic expression only works if the x^2 -term has a coefficient of 1. It's a little more complicated when the x^2 -coefficient (a) isn't 1 — but only a little.

You Can Take Out a Common Factor from Each Term

If you see a **common factor** in each term, such as a number or a variable, **take it out** first.

Example 1

Factor $3x^2 + 15x + 18$.

Solution

$$3x^2 + 15x + 18 = 3(x^2 + 5x + 6)$$

The expression in parentheses can be factored using the method in Topic 6.6.1:

$$= 3(x + 2)(x + 3)$$

But — if the expression in parentheses still has $a \neq 1$, then the expression will need to be factored using the second method, shown in Example 2.

✓ Guided Practice

Factor each expression completely.

- | | |
|---------------------------------|------------------------------------|
| 1. $3x^2 + 15x + 12$ | 2. $4y^2 - 12y - 112$ |
| 3. $2t^2 - 22t + 60$ | 4. $3r^2 - 75$ |
| 5. $4x^2 + 32x + 64$ | 6. $7p^2 + 70p + 63$ |
| 7. $5m^2 + 20m + 15$ | 8. $6x^2 + 42x + 60$ |
| 9. $-2k^2 - 20k - 32$ | 10. $-3m^2 - 30m - 72$ |
| 11. $2 + x - x^2$ | 12. $-3x^2 - 84x - 225$ |
| 13. $3x^2y - 33xy - 126y$ | 14. $10x^2 + 290x + 1000$ |
| 15. $100 + 75x - 25x^2$ | 16. $100n^2y + 100ny - 5600y$ |
| 17. $32a^2 - 8x^2a^2$ | 18. $21a - 80 - a^2$ |
| 19. $2x^2m^2 + 28x^2m - 102x^2$ | 20. $3a^2b^2x^2 + 30a^2bx + 63a^2$ |

You Can Factor $ax^2 + bx + c$ by Trial and Error

If you **can't** see a **common factor**, then you need to get $ax^2 + bx + c$ into the form $(a_1x + c_1)(a_2x + c_2)$, where a_1 and a_2 are factors of a , and c_1 and c_2 are factors of c .

$$ax^2 + bx + c = (a_1x + c_1)(a_2x + c_2)$$

$a = a_1 a_2$
 $a_1 c_2 + a_2 c_1 = b$
 $c = c_1 c_2$

Note that if a number is **positive** then its two factors will be either **both positive** or **both negative**.

If a number is **negative**, then its two factors will have **different signs** — **one positive** and **one negative**.

These facts give important clues about the signs of c_1 and c_2 .

Example 2

Factor $3x^2 + 11x + 6$.

Solution

Write down pairs of factors of $a = 3$: $a_1 = 1$ and $a_2 = 3$

Write down pairs of factors of $c = 6$:
 $c_1 = 1$ and $c_2 = 6$
 $c_1 = 2$ and $c_2 = 3$

Now find the combination of factors that gives $a_1c_2 + a_2c_1 = b = 11$.

Put the x -terms into parentheses first, with the pair of coefficients 3 and 1: $(3x \quad)(x \quad)$.

Now try all the pairs of c_1 and c_2 in the parentheses and find the possible values of $a_1c_2 + a_2c_1$ (and $a_1c_2 - a_2c_1$):

$(3x \quad 1)(x \quad 6)$ multiplies to give $18x$ and x , which add/subtract to give $19x$ or $17x$.
 $(3x \quad 6)(x \quad 1)$ multiplies to give $3x$ and $6x$, which add/subtract to give $9x$ or $3x$.
 $(3x \quad 2)(x \quad 3)$ multiplies to give $9x$ and $2x$, which add/subtract to give **$11x$** or $7x$.
 $(3x \quad 3)(x \quad 2)$ multiplies to give $6x$ and $3x$, which add/subtract to give $9x$ or $3x$.

So $(3x \quad 2)(x \quad 3)$ is the combination that gives $11x$ (so $b = 11$).

Now fill in the $+ / -$ signs.

Both c_1 and c_2 are positive (since $c = c_1c_2$ and $b = a_1c_2 + a_2c_1$ are positive), so the final factors are $(3x + 2)(x + 3)$.

Check by expanding the parentheses to make sure they give the original equation:

$$(3x + 2)(x + 3) = 3x^2 + 9x + 2x + 6 = 3x^2 + 11x + 6$$

That's what you started with, so **$(3x + 2)(x + 3)$** is the correct factorization.

Check it out:

Each pair of coefficients c_1 and c_2 has TWO possible positions.

✓ Guided Practice

Factor each polynomial.

- | | |
|------------------------|------------------------|
| 21. $2x^2 + 5x + 2$ | 22. $2a^2 + 13a + 11$ |
| 23. $2y^2 + 7y + 3$ | 24. $4x^2 + 28x + 49$ |
| 25. $4x^2 + 12x + 9$ | 26. $6x^2 + 23x + 7$ |
| 27. $3x^2 + 13x + 12$ | 28. $2x^2 + 11x + 5$ |
| 29. $4a^2 + 16a + 7$ | 30. $2p^2 + 14p + 12$ |
| 31. $8a^2 + 46a + 11$ | 32. $3g^2 + 51g + 216$ |
| 33. $9x^2 + 12x + 4$ | 34. $4a^2 + 36a + 81$ |
| 35. $4b^2 + 32b + 55$ | 36. $3x^2 + 22x + 24$ |
| 37. $10a^2 + 23a + 12$ | 38. $6t^2 + 23t + 20$ |
| 39. $4x^2 + 34x + 16$ | 40. $15b^2 + 96b + 36$ |

Be Careful if There's a Minus Sign

Example 3

Factor $6x^2 + 5x - 6$.

Solution

Write down pairs of factors of $a = 6$: \Rightarrow $6x^2 + 5x - 6$ Write down pairs of factors of $c = -6$ (ignoring the minus sign for now): \Leftarrow

$a_1 = 6$ and $a_2 = 1$	$c_1 = 1$ and $c_2 = 6$
$a_1 = 2$ and $a_2 = 3$	$c_1 = 2$ and $c_2 = 3$

Put the x -terms into parentheses first, with the first pair of possible values for a_1 and a_2 , 6 and 1: $(6x \quad)(1x \quad)$. Now try all the pairs of c_1 and c_2 in the parentheses and find $a_1c_2 + a_2c_1$ and $a_1c_2 - a_2c_1$ like before:

$(6x \quad 1)(x \quad 6)$ multiplies to give $36x$ and x , which add/subtract to give $37x$ or $35x$.
 $(6x \quad 6)(x \quad 1)$ multiplies to give $6x$ and $6x$, which add/subtract to give $12x$ or $0x$.
 $(6x \quad 2)(x \quad 3)$ multiplies to give $18x$ and $2x$, which add/subtract to give $20x$ or $16x$.
 $(6x \quad 3)(x \quad 2)$ multiplies to give $12x$ and $3x$, which add/subtract to give $15x$ or $9x$.

None of these combinations works, so try again with $(2x \quad)(3x \quad)$:

$(2x \quad 1)(3x \quad 6)$ multiplies to give $12x$ and $3x$, which add/subtract to give $15x$ or $9x$.
 $(2x \quad 6)(3x \quad 1)$ multiplies to give $2x$ and $18x$, which add/subtract to give $20x$ or $16x$.
 $(2x \quad 3)(3x \quad 2)$ multiplies to give $4x$ and $9x$, which add/subtract to give $13x$ or $5x$.

$5x$ is what you want, so you can stop there — so $(2x \quad 3)(3x \quad 2)$ is the right combination.

Now fill in the $+/-$ signs to get $b = +5$. One of c_1 and c_2 must be negative, to give $c = -6$, so the final factors are either $(2x + 3)(3x - 2)$ or $(2x - 3)(3x + 2)$.

The x -term of $(2x + 3)(3x - 2)$ will be $9x - 4x = 5x$, whereas the x -term for $(2x - 3)(3x + 2)$ will be $4x - 9x = -5x$.

So the correct factorization is $(2x + 3)(3x - 2)$.

Check it out:

You can consider separately whether the values of c_1 and c_2 should be positive or negative.

✓ Guided Practice

Factor each polynomial.

41. $2x^2 + 3x - 2$

43. $5k^2 + 13k - 6$

45. $6b^2 - 23b + 7$

47. $3k^2 - 2k - 1$

49. $18 + 5x - 2x^2$

51. $9x^2 + 12x + 4$

53. $3x^2 - 7x - 6$

55. $6x^2 + 2x - 20$

57. $6y^2 + y - 12$

42. $3y^2 - y - 2$

44. $3x^2 - x - 10$

46. $2x^2 - 5x + 2$

48. $3v^2 - 16v + 5$

50. $28 + x - 2x^2$

52. $7a^2 - 26a - 8$

54. $12x^2 + 5x - 2$

56. $18x^2 + x - 4$

58. $9m^2 - 3m - 20$

Use the + and – Signs in the Quadratic to Work Faster

Looking carefully at the signs in the quadratic that you are factoring can help to narrow down the choices for a_1 , a_2 , c_1 , and c_2 .

Example 4

Factor $3x^2 + 11x + 6$.

Solution

Here $c = 6$, which is positive — so its factors c_1 and c_2 are either both positive or both negative.

But since $b = 11$ is positive, you can tell that c_1 and c_2 must be positive (so that $a_1c_2 + a_2c_1$ is positive).

Example 5

Factor $3x^2 - 11x + 6$.

Solution

Here c is also positive, so c_1 and c_2 are either both positive or both negative.

But since $b = -11$ is negative, you can tell that c_1 and c_2 must be negative (so that $a_1c_2 + a_2c_1$ is negative).

Example 6

Factor $6x^2 + 5x - 6$.

Solution

In this expression, c is negative, so one of c_1 and c_2 must be positive and the other must be negative.

So instead of looking at both the sums and differences of all the different combinations a_1c_2 and a_2c_1 , you only need to look at the differences.

✓ Guided Practice

Factor each expression.

59. $2n^2 + n - 3$

60. $2x^2 - 5x - 3$

61. $4a^2 + 4a + 1$

62. $3x^2 - 4x + 1$

63. $9y^2 + 6y + 1$

64. $4t^2 + t - 3$

65. $5x^2 - x - 18$

66. $9x^2 - 6x + 1$

67. $6t^2 + t - 1$

68. $b^2 + 10b + 21$

✓ Independent Practice

Factor each polynomial.

1. $5k^2 - 7k + 2$

2. $4k^2 - 15k + 9$

3. $12t^2 - 11t + 2$

4. $9 - 7k - 2k^2$

5. $10 + h - 3h^2$

6. $15x^2 - 14x - 8$

7. $6 + 5x - 6x^2$

8. $3x^4 + 8x^2 - 3$

9. If the area of a rectangle is $(6x^2 + 25x + 14)$ square units and the length is $(3x + 2)$ units, find the width w in terms of x .

10. The area of a parallelogram is $(12x^2 + 7x - 10)$ cm², where x is positive. If the area is given by the formula Area = base \times height, find the base length and the height of the parallelogram, given that they are both linear factors of the area.

Factor each of these expressions completely.

11. $x^2ab - 3abx - 18ab$

12. $(d + 2)x^2 - 7x(d + 2) - 18(d + 2)$

13. $(x + 1)m^2 - 2m(x + 1) + (x + 1)$

14. $-2dk^2 - 14dk + 36d$

15. Five identical rectangular floor tiles have a total area of $(15x^2 + 10x - 40)$ m². Find the dimensions of each floor tile, if the length of each side can be written in the form $ax + b$, where a and b are integers.

Round Up

Now you can *factor* lots of different types of polynomials. In the next Section you'll learn about another type — *quadratic expressions containing two different variables*.

Section 6.7

Factoring Quadratics in Two Variables

California Standards:

11.0: Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

What it means for you:

You'll learn how to factor quadratic expressions containing two variables.

Key words:

- quadratic
- factor
- trial and error

Check it out:

Following the rules from Topic 6.6.2, both of the terms must be positive — so the + signs can be put in the parentheses from the start.

So far, most of the quadratics you've factored have had only one variable — but the same rules apply if there are two variables.

Quadratics with Two Variables Can Also Be Factored

In Section 6.6 you saw that a quadratic expression such as $x^2 + 2x + 1$ can be written as two factors — in this case $(x + 1)(x + 1)$.

The same is true of an expression such as $x^2 + 2xy + y^2$ — it can be written as $(x + y)(x + y)$.

The method for factoring an expression like this is the same as before:

Example 1

Factor the following expression: $x^2 + 4xy + 3y^2$

Solution

$$x^2 + 4xy + 3y^2 = (x + \quad)(x + \quad)$$

To fill the gaps you need two numbers or expressions that will multiply together to make $3y^2$ and add together to make $4y$.

Try out some sets of numbers or expressions that multiply to make $3y^2$:

- $3y^2$ and 1 add together to make $3y^2 + 1$
- $3y$ and y add together to make $4y$

So $x^2 + 4xy + 3y^2 = (x + 3y)(x + y)$.

Example 2

Factor the following expression: $3p^2 + 5pq + 2q^2$

Solution

$$3p^2 + 5pq + 2q^2 = (3p + \quad)(p + \quad)$$

Filling these gaps is a little more complicated. You need two numbers or expressions that will multiply together to make $2q^2$ and, when multiplied by the $3p$ and p respectively, add together to make $5pq$.

Try out some sets of numbers or expressions:

- $(3p + q)(p + 2q)$ — this would give pq -terms of $6pq$ and pq , which add to make $7pq$
- $(3p + 2q)(p + q)$ — this would give pq -terms of $3pq$ and $2pq$, which add to make $5pq$

$5pq$ is what you need, so $3p^2 + 5pq + 2q^2 = (3p + 2q)(p + q)$.

✓ Guided Practice

Factor each polynomial below.

1. $x^2 + 3xy + 2y^2$

3. $a^2 + 9ab + 18b^2$

5. $d^2 + 21md + 20m^2$

7. $x^2 - xy - 2y^2$

9. $a^2 - ab - 12b^2$

2. $x^2 + 14xy + 40y^2$

4. $p^2 + 7pq + 12q^2$

6. $k^2 + 8pk + 12p^2$

8. $x^2 - 2xz - 8z^2$

10. $x^2 - 5xy + 6y^2$

The areas of the rectangles below are the products of two binomials with integer coefficients. Find the possible length and width of each rectangle.

11. Area = $(x^2 + 3xa + 2a^2)$ ft²

12. Area = $(y^2 - 9yb + 14b^2)$ in²

13. Area = $(3x^2 - 4xb + b^2)$ ft²

14. Area = $(5a^2 - ab - 18b^2)$ m²

15. Area = $(4c^2 + 4cd + d^2)$ ft²

16. Area = $(12x^2 - 5xy - 2y^2)$ in²

Trial and Error is Tricky if There are More Possibilities

Example 3

Factor the following expression: $2m^2 - 11mp + 5p^2$

Solution

$2m^2$ can be factored into $2m$ and m , so:

$$2m^2 - 11mp + 5p^2 = (2m - \quad)(m - \quad)$$

To fill the gaps you need two terms in p that will multiply together to make $5p^2$, and when multiplied by $2m$ and m respectively, will add together to make $-11mp$.

Try out some sets of parentheses that multiply to make $5p^2$:

- $(2m - 5p)(m - p)$ — this would give mp -terms of $-2mp$ and $-5mp$, which add to make $-7mp$
- $(2m - p)(m - 5p)$ — this would give mp -terms of $-10mp$ and $-mp$, which add to make $-11mp$

The second one gives the $-11mp$ needed, so $2m^2 - 11mp + 5p^2 = (2m - p)(m - 5p)$.

Check it out:

Both p -terms must be negative, so that the mp -terms add up to a negative term and the p -terms multiply to give a positive p^2 -term.

Example 4

Factor the following expression: $9x^2 + 6xz - 8z^2$

Solution

The z^2 -term is negative, so one of the z -terms will be negative and the other positive. So that means there are a lot more combinations to try out.

$9x^2$ can be made by either $3x \times 3x$ or $9x \times x$, and $-8z^2$ can be made by any of $-2z \times 4z$, $2z \times -4z$, $8z \times -z$, and $-8z \times z$.

Try out some sets of parentheses:

- $(9x - 4z)(x + 2z)$ — this would give $18xz$ and $-4xz$, which add to make $+14xz$
- $(9x + 2z)(x - 4z)$ — this would give $-36xz$ and $2xz$, which add to make $-34xz$
- $(9x - 2z)(x + 4z)$ — this would give $36xz$ and $-2xz$, which add to make $+34xz$
- $(9x + 4z)(x - 2z)$ — this would give $-18xz$ and $4xz$, which add to make $-14xz$
- $(3x - 4z)(3x + 2z)$ — this would give $6xz$ and $-12xz$, which add to make $-6xz$
- $(3x - 2z)(3x + 4z)$ — this would give $12xz$ and $-6xz$, which add to make **$+6xz$**

You can stop here because $+6xz$ is the expression you are trying to get. So $9x^2 + 6xz - 8z^2 = (3x - 2z)(3x + 4z)$.

✓ Guided Practice

Factor each of the polynomials below.

17. $2x^2 - 5xy - 3y^2$

18. $3m^2 - 7mp + 2p^2$

19. $3x^2 + 17xy + 10y^2$

20. $4x^2 + 9xy + 2y^2$

21. $3g^2 + 7gh + 4h^2$

22. $2a^2 + 9ab + 9b^2$

23. $4f^2 - 16gf + 15g^2$

24. $49w^2 + 7wz - 6z^2$

25. $8m^2 - 2mh - 15h^2$

26. $6x^2 + 17xy + 12y^2$

The areas of the rectangles below are the products of two binomials with integer coefficients. Find the possible dimensions of the rectangles.

27. Area = $(2m^2 + 3mn - 2n^2)$ ft²

28. Area = $(a^2 + 8ab + 15b^2)$ in.²

29. Area = $(3x^2 + 10xy + 8y^2)$ m²

30. Area = $(15x^2 - 29xy + 14y^2)$ ft²

31. Area = $(6a^2 + 11ab - 10b^2)$ ft²

32. Area = $(6c^2 + 10cd + 4d^2)$ m²

✓ Independent Practice

Factor each of these polynomials.

- $p^2 - 7pq + 10q^2$
- $g^2 + 4hg - 21h^2$
- $c^2 + 3cd - 40d^2$
- $m^2 + 8mn - 20n^2$
- $y^2 - 8xy + 15x^2$
- $5p^2 + 26pq + 5q^2$
- $2r^2 + 11rk + 14k^2$
- $3x^2 - 8xy - 3y^2$

Simplify and factor the following polynomials.

- $(x^2 + 13x + 8) - (3x^2 + 10x - 3) - (x^2 + 2x + 6) + (4x^2 - 5x - 2)$
- $(5x^2 + 2x + 4) - (6x^2 - 3x + 7) + (4x^2 - x - 4)$
- $(4x^2 - 6xy - 10y^2) - (2x^2 - 8xy + 2y^2)$
- $(6x^2 + 3xy + 8y^2) - (3x^2 - 12xy - 10y^2)$
- $(2t^2 - 8tz - 5z^2) - (4t^2 + 2tz - 15z^2) + (5t^2 + tz - 40z^2)$
- $(6x^2 - 4xy - 25y^2) - (2x^2 + 4xy + 25y^2) - (2x^2 + 4xy - 18y^2)$

The areas of the circles below are products of π and a binomial squared. Find the radius of each circle.

- Area of circle is $(9\pi x^2 + 24\pi xy + 16\pi y^2)$ in.²
- Area of the circle is $(4\pi x^2 - 20\pi ax + 25\pi a^2)$ m²
- Area of the circle is $(9\pi y^2 + 48\pi zy + 64\pi z^2)$ ft²
- Area of the circle is $(4\pi m^2 + 20\pi mn + 25\pi n^2)$ ft²

The areas of the parallelograms below are the products of two binomials with integer coefficients. Find the dimensions of each parallelogram if $a = 10$ and $b = 5$.

- Parallelogram with area $(8a^2 - 10ab + 3b^2)$ ft²
- Parallelogram with area $(4a^2 - 9b^2)$ ft²
- Parallelogram with area $(12a^2 + 11ab - 5b^2)$ ft²
- Parallelogram with area $(40a^2 - 51ab - 7b^2)$ ft²

Factor the following polynomials.

- $6a^2z^2k + 20z^2abk + 16b^2z^2k$
- $36a^2b^2c - 15ab^3c - 6b^4c$
- $25x^3y^2 - 5x^2y^3 - 90xy^4$
- $16a^2z^2c + 16abz^2c + 4b^2z^2c$

Factor and simplify completely.

- $12x^2(x + 2) + 25xy(x + 2) + 12y^2(x + 2)$
- $18a^2b^2(a - 1) - 33ab^3(a - 1) - 30b^4(a - 1)$

Round Up

*This is a long process, so it's easy to make mistakes. You should always **check** your answer by multiplying out the parentheses again. If you don't get the expression you started with, you must have gone wrong somewhere. That means you'll need to go back a stage in your work and try a different combination of factors.*

Factoring Third-Degree Polynomials

California Standards:

11.0: Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

What it means for you:

You'll learn how to factor third-degree expressions.

Key words:

- polynomial
- degree
- factor

Don't forget:

See Topic 6.6.2 for more on common factors.

Quadratics are second-degree polynomials because they have an x^2 -term. Now you'll factor polynomials with an x^3 -term too.

Factor Third-Degree Polynomials in Stages

To factor a **third-degree polynomial**, the first thing you should do is look for a common factor.

Separate any obvious common factors, then try to factor the remaining expression (the part inside the parentheses).

Example 1

Factor $x^3 + 7x^2 + 12x$ completely.

Solution

All of the terms in $x^3 + 7x^2 + 12x$ contain x , so x is a factor:

$$\begin{aligned} x^3 + 7x^2 + 12x &= x \left(\frac{x^3}{x} + \frac{7x^2}{x} + \frac{12x}{x} \right) \\ &= x(x^2 + 7x + 12) \end{aligned}$$

Now look at the factor in the parentheses — this is a quadratic expression that it may be possible to factor.

In this case it's possible to factor it, using the method from Section 6.6:

$$\begin{aligned} x^3 + 7x^2 + 12x &= x(x^2 + 7x + 12) \\ &= \mathbf{x(x + 3)(x + 4)} \end{aligned}$$

✓ Guided Practice

Factor completely these polynomials.

- | | |
|--|---|
| 1. $4y^3 + 26y^2 + 40y$ | 2. $24x^3 - 33x^2 + 9x$ |
| 3. $6x^3 - 7x^2 - 20x$ | 4. $12x^3 - 18x^2 - 12x$ |
| 5. $6a^3b^2 + 33a^2b^2 + 15ab^2$ | 6. $6k^3j^3 - 10k^2j^3 - 4kj^3$ |
| 7. $10a^3b^2c^2 + 45a^2b^2c^2 + 20ab^2c^2$ | 8. $16x^4z^2 + 12x^3z^2 - 4x^2z^2$ |
| 9. $12b^4y + 38b^3y + 30b^2y$ | 10. $24c^5t^2 + 132c^4t^2 + 144c^3t^2$ |
| 11. $36h^3k^4 + 12h^2k^4 - 63hk^4$ | 12. $40x^6y^5 + 11x^5y^5 - 2x^4y^5$ |
| 13. $189b^6c^4 - 60b^5c^4 - 96b^4c^4$ | 14. $24x^2b^2c^2 + 44xb^2c^2 - 140b^2c^2$ |
| 15. $12x^4y^2 + 78x^3y^2 + 108x^2y^2$ | 16. $12a^4b^2 + 46a^3b^2 + 40a^2b^2$ |
| 17. $18b^4c - 87b^3c + 105b^2c$ | 18. $12a^4b^2c - 46a^3b^2c + 40a^2b^2c$ |

Example 2

Factor $-2x^3 - 2x^2 + 4x$ completely.

Solution

- Again, x is a factor.
- Each term has an even coefficient — so you can take out a factor of 2.
- And given that two of the terms are negative, you can take out a factor of -2 instead of 2.

This is helpful because it means that the coefficient of x^2 becomes 1 — which makes the quadratic expression much easier to factor, as you saw in Section 6.6.

$$\begin{aligned} -2x^3 - 2x^2 + 4x &= -2x \left(\frac{-2x^3}{-2x} + \frac{-2x^2}{-2x} + \frac{4x}{-2x} \right) \\ &= -2x(x^2 + x - 2) \end{aligned}$$

Now factor the quadratic:

$$\begin{aligned} -2x^3 - 2x^2 + 4x &= -2x(x^2 + x - 2) \\ &= \mathbf{-2x(x - 1)(x + 2)} \end{aligned}$$

Note that in Example 2, if $2x$ had been factored out instead of $-2x$, the result would have been:

$$\begin{aligned} -2x^3 - 2x^2 + 4x &= 2x \left(\frac{-2x^3}{2x} + \frac{-2x^2}{2x} + \frac{4x}{2x} \right) \\ &= 2x(-x^2 - x + 2) \\ &= 2x(1 - x)(x + 2) \quad \text{or} \quad 2x(x - 1)(-x - 2) \end{aligned}$$

...which is also correct, but is a little trickier to factor.

Guided Practice

Factor completely the polynomials below.

- | | |
|-------------------------------------|--|
| 19. $-4x^2y^2 - 6xy^2 + 4y^2$ | 20. $-8xy^4 + 4xy^3 + 60xy^2$ |
| 21. $-12a^3b^3 - 30a^3b^2 + 18a^3b$ | 22. $-8w^3k - 42w^2k - 10wk$ |
| 23. $-4a^3b - 26a^2b - 36ab$ | 24. $-90b^4c^2 - 174b^3c^2 - 48b^2c^2$ |
| 25. $-16a^3b^2 + 16ab^2$ | 26. $-45x^4y^2 - 50x^3y^2 - 5x^2y^2$ |
| 27. $-147x^3y + 84x^2y - 12xy$ | 28. $-162c^5d + 288c^4d - 128c^3d$ |
| 29. $18b^4c - 66b^3c + 60b^2c$ | 30. $-50y^4z - 130y^3z + 60y^2z$ |
| 31. $-42c^4d - 28c^3d + 14c^2d$ | 32. $-54y^3z^2 + 21y^2z^2 + 3yz^2$ |
| 33. $-162a^2b^3 + 2a^2b$ | 34. $16a^4b^2 + 176a^3b^2 + 484a^2b^2$ |
| 35. $-60b^2d^6 + 9b^2d^5 + 6b^2d^4$ | 36. $-12b^4f^2 + 68b^3f^2 - 96b^2f^2$ |

Independent Practice

Factor these polynomials completely.

- $-8a^3 + 78a^2 + 20a$
- $18x^2y^2 + 51xy^2 + 15y^2$
- $14y^2d^2 + 7yd^2 - 42d^2$
- $30x^3y^3 - 35x^2y^3 - 100xy^3$
- $250x^2y^4 + 100x^2y^3 + 10x^2y^2$
- $120a^3b^3c^3 + 28a^2b^3c^3 - 8ab^3c^3$

7. Which of the following is equivalent to $18a^2x^2 + 3a^2x - 3a^2$?

- (i) $3a^2(2x + 1)(2x - 1)$ (ii) $3a^2(3x - 1)(2x + 1)$

8. Which of the following is equivalent to $18a^2x^2 - 32a^2$?

- (i) $2a^2(3x + 4)(3x - 4)$ (ii) $2a^2(3x - 4)(3x - 4)$

9. Which of the following is a factor of $35x^2 + 64ax + 21a^2$?

- (i) $5x + 3a$ (ii) $5x + 7a$ (iii) $7x + 5a$ (iv) $7x + 7a$

10. Which of the following is a factor of $8y^2a^2 - 50ya^3 - 42a^4$?

- (i) $2a^3$ (ii) $8a^2$ (iii) $4y - 7a$ (iv) $4y + 3a$

Find the value of ? in the problems below. (The symbol “ \equiv ” means that the equation is true for all values of the variables.)

11. $81j^4 - 36j^2 + 4j^2 \equiv j^2(9j - 2)^2$

12. $18w^3 - 48w^2 + 32w \equiv 2w(3w - 4)^2$

13. $12a^3b + 70a^2b + 72ab \equiv 2ab(2a + 9)(3a + 4)$

14. $-12a^4b^2 - 58a^3b^3 - 70a^2b^3 \equiv -2a^2b^3(2a + 5)(3a + 7)$

15. $-56a^4b^4 + 12a^3b^4 + 8a^2b^4 \equiv -?a^2b^4(2a - 1)(7a + 2)$

16. A cylinder has a base with dimensions that are binomial factors. If the volume of the cylinder is $(75\pi x^3 + 30\pi x^2 + 12\pi x)$ in.³ and the height is $3x$ in., find the radius of the base, in terms of x . ($V = \pi r^2 h$)

17. A rectangular prism has a base with dimensions that are binomial factors. If the volume of the prism is $(30x^3 - 28x^2 - 16x)$ in³ and the height is $2x$ in., find the dimensions of the base, in terms of x .

18. The product of three consecutive odd integers is $x^3 + 6x^2 + 8x$. Find each of the three integers in terms of x .

19. The volume of a rectangular box is $(6x^3 + 17x^2 + 7x)$ cubic inches, and its height is x inches. Find the dimensions of the base of the box, w and l , in terms of x , given that w and l can be expressed in the form $(ax + b)$, where a and b are integers.

20. A cylinder has a base with dimensions that are binomial factors. If the volume of the cylinder is $(36\pi x^3 - 96\pi x^2 + 64\pi x)$ ft³ and the height is $4x$ ft, find the radius of the base, in terms of x . ($V = \pi r^2 h$)

Round Up

Take care when you're factoring cubic expressions — you need to take it step by step.

Often you'll be able to factor out a term containing x — then you'll be left with a normal quadratic inside the parentheses. Look back at Section 6.6 if you're having trouble with the quadratic part.

Topic 6.8.1

Section 6.8

The Difference of Two Squares

California Standards:

11.0: Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

What it means for you:

You'll use the difference of two squares to factor quadratics.

Key words:

- difference of two squares
- quadratic

Don't forget:

The difference of two squares equation was derived in Topic 6.4.1.

Being able to recognize the difference of two squares is really useful — it helps you factor quadratic expressions.

Use Difference of Two Squares to Factor Quadratics

A difference of two squares is one term squared minus another term squared: $m^2 - c^2$.

You can use this equation to factor the difference of two squares:

$$m^2 - c^2 = (m + c)(m - c)$$

$$\begin{array}{l} \text{The difference} \\ \text{of two squares} \\ m^2 - c^2 \end{array} = \begin{array}{l} \text{The sum of} \\ \text{the two terms} \\ (m + c) \end{array} \times \begin{array}{l} \text{The difference} \\ \text{between the two terms} \\ (m - c) \end{array}$$

Example 1

Factor $x^2 - 9$.

Solution

Substitute x^2 for m^2 and 9 for c^2 in the difference of two squares equation and you get:

$$x^2 - 9 = (x + 3)(x - 3)$$

The square root of 9 is 3. Don't forget the opposite signs.

✓ Guided Practice

Factor each expression completely.

1. $x^2 - 16$
2. $a^2 - 25$
3. $c^2 - 49$
4. $a^2 - 100$
5. $x^2 - y^2$
6. $81 - x^2$
7. $64 - c^2$
8. $144 - y^2$
9. $11x^2 - 176$
10. $3y^2 - 300$
11. $7x^2 - 63$
12. $5a^2 - 125$
13. $3m^2n - 12n$
14. $6a^3 - 216a$
15. $162a - 2a^3$
16. $7x^3y - 7xy^3$

Work Out the Square Root of Each Term

Example 2

Factor $4x^2 - 25b^2$.

Solution

$4x^2 = (2x)^2$, so the square root of $4x^2$ is $2x$.

$25b^2 = (5b)^2$, so the square root of $25b^2$ is $5b$.

Put the values into the difference of two squares equation:

$$m^2 - c^2 = (m + c)(m - c)$$

$$4x^2 - 25b^2 = (2x + 5b)(2x - 5b)$$

Square root of $4x^2$
Square root of $25b^2$

✓ Guided Practice

Factor each expression completely.

17. $a^2 - 4b^2$

18. $4y^2 - 81x^2$

19. $9a^2 - 64x^4$

20. $4a^2 - 16x^2$

21. $49x^2 - 4y^2$

22. $25y^4 - 81c^8$

23. $2c^3 - 98c$

24. $3m^2n^3 - 12a^2n$

25. $36x^2b^2 - 49a^2c^2$

26. $16xbc^2 - xba^2$

27. $75m^3n^2 - 108a^2b^2m$

28. $18a^2b - 242c^2b$

✓ Independent Practice

Factor each polynomial completely.

1. $5m^3n - 80mn$

2. $54a^3b - 24ab$

3. $343m^3 - 252mn^2$

4. $50b^3 - 18bc^2$

5. $3a^3b - 3ab^3$

6. $(2x + 1)x^2 - (2x + 1)$

7. $4(3x + 2)a^2b^2 - 9(3x + 2)$

8. $4a^2b^2 - 100b^2$

9. $3(9a^2 - 64b^2)a + 8b(9a^2 - 64b^2)$

10. $(2x + 5b)4x^2 - (2x + 5b)25b^2$

The areas of the rectangles below are the products of two binomials. Find the two binomials.

11. Area = $(81x^2 - 100y^2)$ ft²

12. Area = $(16a^2 - 9y^2)$ ft²

13. Area = $(100a^2 - b^2)$ ft²

14. Area = $(4a^2b^2 - 25x^2)$ cm²

15. A triangle has area $\frac{1}{2}(x^2 - 8b^2)$ in.². Find expressions for its base and height dimensions, given that they are binomial factors of the area. Find the base and height when $x = 10$ and $b = 2$, if the height is greater than the length.

Round Up

If you see any quadratic expression in the form $m^2 - c^2$, you can use the *difference of two squares* to factor it as $(m + c)(m - c)$, without needing to do all the math.

Perfect Square Trinomials

California Standards:

11.0: Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and **recognizing perfect squares of binomials.**

What it means for you:

You'll learn about how to factor special quadratics called perfect square trinomials.

Key words:

- perfect square trinomial
- quadratic
- binomial
- factor

Don't forget:

If this seems a bit unfamiliar, take a look at Topic 6.4.1 on special binomial products.

Perfect square trinomials are quadratic expressions of the form $(m + c)^2$ or $(m - c)^2$.

The Square of a Binomial is a Perfect Square Trinomial

You can use one of two equations to work out the square of a binomial:

$$(m + c)^2 = m^2 + 2mc + c^2$$

Square of binomial	=	First term squared	+	Twice the product of both terms	+	Second term squared
$(m + c)^2$		m^2		$2mc$		c^2

Or, if the second term in the binomial is subtracted:

$$(m - c)^2 = m^2 - 2mc + c^2$$

Square of binomial	=	First term squared	-	Twice the product of both terms	+	Second term squared
$(m - c)^2$		m^2		$2mc$		c^2

Use the Equations to Factor Perfect Square Trinomials

Example 1

Factor $x^2 + 2xy + y^2$.

Solution

Substitute x^2 for m^2 , $2xy$ for $2mc$, and y^2 for c^2 in the **first** equation above:

$$\begin{aligned} (m + c)^2 &= m^2 + 2mc + c^2 \\ \Rightarrow x^2 + 2xy + y^2 &= (x + y)^2 \end{aligned}$$

Sometimes you need to factor each term in the expression to get it into the correct form.

Example 2Factor $4x^2 - 12xy + 9y^2$.**Solution**

Factor each term:

$$\begin{array}{c}
 4x^2 - 12xy + 9y^2 \\
 \swarrow \quad \uparrow \quad \nwarrow \\
 4x^2 = (2x)^2 \quad 12xy = 2(2x \cdot 3y) \quad 9y^2 = (3y)^2
 \end{array}$$

$$\text{So: } 4x^2 - 12xy + 9y^2 = (2x)^2 - 2(2x \cdot 3y) + (3y)^2$$

Substitute $(2x)^2$ for m^2 , $2(2x \cdot 3y)$ for $2mc$, and $(3y)^2$ for c^2 in the **second** perfect square trinomial equation:

$$(m - c)^2 = m^2 - 2mc + c^2$$

$$\Rightarrow (2x)^2 - 2(2x \cdot 3y) + (3y)^2 = \mathbf{(2x - 3y)^2}$$

✓ Guided Practice

Factor each expression completely.

- | | |
|--------------------------|---------------------------|
| 1. $m^2 + 2m + 1$ | 2. $4r^2 - 4ry + y^2$ |
| 3. $25y^2 + 10y + 1$ | 4. $9k^2 + 6ky + y^2$ |
| 5. $9x^2 - 6x + 1$ | 6. $k^2 + 4k + 4$ |
| 7. $16x^2 - 24x + 9$ | 8. $4r^2x^2 + 4rkx + k^2$ |
| 9. $25k^2 - 20kt + 4t^2$ | 10. $m^2r^2 - 6mr + 9$ |

✓ Independent Practice

Factor each expression completely.

- | | |
|--------------------------------|--------------------------------|
| 1. $16a^2b^2 + 24ab + 9$ | 2. $25x^2y^2 - 40xya + 16a^2$ |
| 3. $9c^2d^2 + 36cd + 36$ | 4. $49m^2n^2 - 70mn + 25$ |
| 5. $162x^2y^2m + 144xym + 32m$ | 6. $108k^3z^2 - 180k^2z + 75k$ |

Find the radius of each of the circles below, given that the area, A , is the product of a binomial squared and π .

- | | |
|--|---|
| 7. $A = (\pi a^2 + 2\pi a + \pi) \text{ ft}^2$ | 8. $A = (25\pi y^2 + 30\pi y + 9\pi) \text{ ft}^2$ |
| 9. $A = (49\pi m^2 - 56\pi mn + 16\pi n^2) \text{ ft}^2$ | 10. $A = (81\pi x^2y^2 - 90\pi xyz + 25\pi z^2) \text{ ft}^2$ |

The volume, V , of each cylinder below is the product of the height, π , and the radius squared. Find the radius in each case:

11. $V = (98x^2\pi + 84x\pi + 18\pi) \text{ cm}^3$, height = 2 cm
12. $V = (147\pi a^2 - 84\pi ab + 12\pi b^2) \text{ m}^3$, height = 3 m
13. $V = (36\pi x^3 + 48\pi x^2m + 16\pi xm^2) \text{ m}^3$, height = $4x$ m
14. $V = (243x^4\pi - 270\pi x^3b + 75x^2b^2) \text{ ft}^3$, height = $3x^2$ ft

Round Up

The phrase “*perfect square trinomials*” makes this Topic sound much harder than it actually is. They’re really just a special case of the *normal quadratic equations* that you know and love.

Factoring by Grouping

California Standards:

11.0: Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

What it means for you:

You'll group like terms to factor polynomials.

Key words:

- common factor
- like terms

Check it out:

It's difficult to see any common factors in the expression given in Example 1, so it's a good idea to group terms together as a first step.

Grouping like terms means that you can more easily see whether there are common factors in a polynomial — then you can factor them out.

Group Like Terms to See Common Factors

Sometimes you need to group terms together before you can see any **common factors** in an expression — then you can use the **distributive property**, $ab + ac = a(b + c)$, to factor them out.

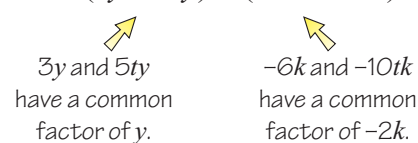
Example 1

Factor by grouping $3y + 5ty - 6k - 10tk$.

Solution

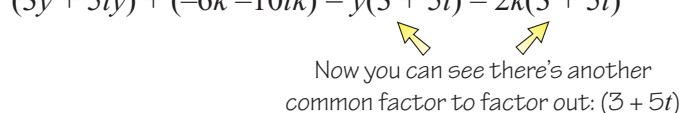
Group $3y + 5ty$ and $-6k - 10tk$ together in parentheses:

$$(3y + 5ty) + (-6k - 10tk)$$



Factor out the common factors:

$$(3y + 5ty) + (-6k - 10tk) = y(3 + 5t) - 2k(3 + 5t)$$



Using the distributive property:

$$y(3 + 5t) - 2k(3 + 5t) = (y - 2k)(3 + 5t)$$

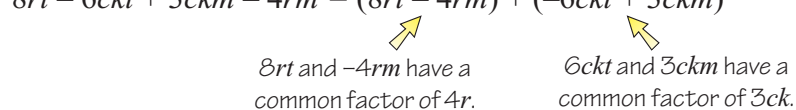
Example 2

Factor completely $8rt - 6ckt + 3ckm - 4rm$.

Solution

Rearrange the expression and group in parentheses:


$$8rt - 6ckt + 3ckm - 4rm = (8rt - 4rm) + (-6ckt + 3ckm)$$



Example 2 continued

Take out the common factors:

$$(8rt - 4rm) + (-6ckt + 3ckm) = 4r(2t - m) + 3ck(-2t + m) \\ = 4r(2t - m) - 3ck(2t - m)$$


Another common factor
to factor out is $(2t - m)$.

Using the distributive property:

$$4r(2t - m) - 3ck(2t - m) = (4r - 3ck)(2t - m)$$

✓ Guided Practice

Factor each expression by grouping.

- $km + 2k - 2m - 4$
- $tx - ty - mx + my$
- $3x + 9 - 4kx - 12k$
- $1 + 3y - 5k - 15ky$
- $1 - k + t - tk$
- $kt - 2k + 3t - 6$
- $6ky + 15t - 10y - 9kt$
- $8hx + 10h - 12tx - 15t$
- $6rx^2 + 15xy - 2rxy - 5y^2$
- $10tx - 3k - 15t + 2kx$

✓ Independent Practice

Factor each expression completely.

- $2x^2 + x + 8x + 4$
- $6x^2 + 9x + 4x + 6$
- $4x^2 + 14x + 14x + 49$
- $2x^2 + 5x + 6x + 15$
- $12n^2 + 21n + 8n + 14$
- $6x^2 + 8x + 15x + 20$
- $n^2 - 16n + 20n - 320$
- $3c^2 - c + 6c - 2$
- $2x^2 + 5xy + 4xy + 10y^2$
- $3m^2 + 3mn - mn - n^2$
- $12a^2 + 9ab - 28ab - 21b^2$
- $2x^3 + 2x^2y + 3xy^2 + 3y^3$
- $4a^2 - 6ab + 6ab - 9b^2$
- $4b^2 - 20bx - 2xb + 10x^2$

Find a value of ? so that the expression will factor into two binomials.

- $20n^2 - 25n + ?n - 20$
- $8xy - 4xz + 4wy - ?wz$
- $?c^2 - 12c + 2c - 4$
- $3a^2 - ?a + 6a - 2$

19. The area of a rectangle is the product of two binomials (with integer coefficients). If the area of the rectangle is $(3a^2 + a + 3a + 1) \text{ m}^2$, find the dimensions of the rectangle.

20. The area of a square is the square of a binomial. If the area of the square is $(4x^2 + 2x + 2x + 1) \text{ in.}^2$, find the side length of the square.

21. The area of a circle is the product of π and the radius squared. If the radius is a binomial and the area of the circle is $(9\pi x^2 + 15\pi xb + 15\pi xb + 25\pi b^2) \text{ in.}^2$, find the radius.

Round Up

That's the end of a Section full of neat little ways of making math a lot less painful. You'll often need to use the methods for *difference of two squares*, *perfect square trinomials*, and *factoring by grouping* — so look back over the Topics in this Section to make sure you understand them.

Chapter 6 Investigation

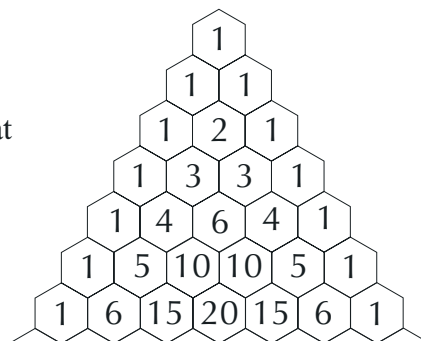
Pascal's triangle

Pascal's triangle was originally developed by the ancient Chinese. However, the French mathematician Blaise Pascal was the first person to discover the importance of all the patterns it contains.

Part 1:

Look at the numbers in the triangle. Write down a rule that could be used to predict each number in the triangle.

Use your rule to predict the next row of the triangle.



Part 2:

Find and simplify the following:

- $(x + 1)^2$
- $(x + 1)^3$
- $(x + 1)^4$

Hint: this is just $(x + 1)(x + 1)^2$

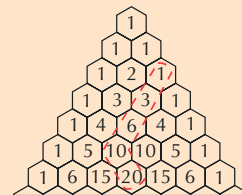
How are your answers linked to Pascal's Triangle?

Use the triangle to predict the expansion of $(x + 1)^5$. Test your prediction.

Extension

Investigate other patterns in Pascal's Triangle. Here are some ideas:

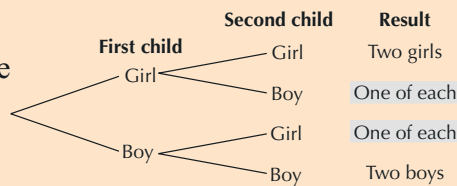
- Look at the numbers in the diagonal lines of the triangle.
- Investigate "hockey stick" shapes, like the one shown on the right.
- Find the sum of each row of numbers. What do you notice?
- Find rows in which the second number is prime. What is special about these rows?



Open-ended Extension

If there are two children in a family, there can either be two girls, two boys, or a girl and a boy.

The probability of each combination and the ratios of the probabilities are shown in the table:



Combination	Probability	Ratio of probabilities
2 girls	$\frac{1}{4}$	1
1 girl, 1 boy	$\frac{1}{2}$	2
2 boys	$\frac{1}{4}$	1

Investigate the link between Pascal's Triangle and the probabilities of having different combinations of boys and girls in families with different numbers of children. Remember, it doesn't matter which order the boys and girls are born in.

Round Up

Although it just looks like a funny pile of shapes and numbers, there are a lot of real-life problems that can be solved using the patterns in Pascal's Triangle.

Chapter 7

Quadratic Equations and Their Applications

Section 7.1	Solving Quadratic Equations	333
Section 7.2	Completing the Square	342
Section 7.3	The Quadratic Formula and Applications	355
Section 7.4	Quadratic Graphs	362
Section 7.5	The Discriminant	372
Section 7.6	Motion Tasks and Other Applications	379
Investigation	The Handshake Problem.....	386

Section 7.1

Solving Quadratic Equations by Factoring

California Standards:

11.0: Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

14.0: Students solve a quadratic equation by factoring or completing the square.

What it means for you:

You'll solve quadratic equations by factoring.

Key words:

- quadratic
- factor
- zero property

In this Topic you'll use all the factoring methods that you learned in Chapter 6 to solve quadratic equations.

Quadratic Equations Have Degree 2

Quadratic equations contain a **squared variable**, but no higher powers — they have **degree 2**.

These are all quadratic equations, as the highest power of the variable is 2:

- (i) $x^2 - 3x + 2 = 0$
- (ii) $4x^2 + 12x - 320 = 0$
- (iii) $y^2 + 4y - 7 = 2y^2 - 2y$

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a , b , and c are numbers, and a is not 0.

For example, in (i) above, $a = 1$, $b = -3$, and $c = 2$, while in (ii), $a = 4$, $b = 12$, and $c = -320$.

Example (iii) above is a quadratic in y , while the others are quadratics in x .

✓ Guided Practice

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a , b , and c are numbers. Identify a , b , and c in these equations.

- | | |
|---|--------------------------------|
| 1. $-x^2 + 5x - 6 = 0$ | 2. $6x^2 + 31x + 35 = 0$ |
| 3. $4x^2 - 12x + 9 = 0$ | 4. $16x^2 - 8x + 1 = 0$ |
| 5. $-x^2 - 4x - 4 = 0$ | 6. $64x^2 + 48x + 9 = 0$ |
| 7. $6y^2 + 28y + 20 = 5 - 6y^2$ | 8. $4x^2 + 6x + 1 = 3x^2 + 8x$ |
| 9. $4(x^2 - 5x) = -25$ | 10. $3x(3x + 4) + 8 = 4$ |
| 11. $7x(7x + 4) + 4x^3 + 3 = 2(2x^3 + 1)$ | |

Solving is Finding Values That Make the Equality True

An **equation** is a statement saying that two mathematical expressions are **equal**.

For example, $7 + 2 = 9$, $4x + 2 = 14$, and $x^2 - 3x + 2 = 0$ are all equations.

If an equation contains a **variable** (an unknown quantity), then **solving** the equation means finding possible values of the variable that make the equation a true statement.

Example 1

Find a solution of the equation $x^2 - 3x + 2 = 0$.

Solution

If you evaluate the above equation using, say, $x = 3$, then you get:

$$3^2 - (3 \times 3) + 2 = 0$$

This is **not a true statement** (since the left-hand side equals 2).

So $x = 3$ is not a solution of the equation.

But if instead you substitute $x = 1$, then you get:

$$1^2 - (3 \times 1) + 2 = 0$$

This is a **true statement**.

So **$x = 1$** is a solution of the equation $x^2 - 3x + 2 = 0$.

✓ Guided Practice

Determine which of the two values given is a solution of the equation.

12. $-x^2 + 5x - 6 = 0$ for $x = -3$ and $x = 3$

13. $6x^2 + 31x + 35 = 0$ for $x = -\frac{5}{3}$ and $x = \frac{5}{3}$

14. $4x^2 - 12x + 9 = 0$ for $x = \frac{1}{2}$ and $x = \frac{3}{2}$

15. $16x^2 - 8x + 1 = 0$ for $x = -\frac{1}{4}$ and $x = \frac{1}{4}$

16. $-x^2 - 4x - 4 = 0$ for $x = 2$ and $x = -2$

17. $64x^2 + 48x + 9 = 0$ for $x = -\frac{3}{8}$ and $x = \frac{3}{8}$

Zero Property — if $xy = 0$, then $x = 0$ or $y = 0$ (or Both)

One way to solve a quadratic equation is to **factor** it and then make use of the following property of **zero**:

Check it out:

This says that if the product of two expressions is zero (so you get zero when you multiply them together), then at least one of those expressions must itself be zero.

Zero Property

If the product $mc = 0$, then either:

(i) $m = 0$,

(ii) $c = 0$,

(iii) both $m = 0$ and $c = 0$.

Example 2

Solve $x^2 + 2x - 15 = 0$ by factoring.

Solution

$$x^2 + 2x - 15 = (x - 3)(x + 5)$$

So if $(x - 3)(x + 5) = 0$, then by the zero property, either $(x - 3) = 0$ or $(x + 5) = 0$.

So either **$x = 3$** or **$x = -5$** .

Example 3

Solve $2x^2 + 3x - 20 = 0$ by factoring.

Solution

$$2x^2 + 3x - 20 = (2x - 5)(x + 4) = 0$$

So either $x = \frac{5}{2}$ or $x = -4$.

✓ Guided Practice

Solve each of these quadratic equations by using the zero property.

18. $(2x + 7)(3x + 5) = 0$

19. $(x - 5)(x - 1) = 0$

20. $49x^2 - 1 = 0$

21. $64a^2 - 25 = 0$

22. $4x^2 + 8x + 3 = 0$

23. $2x^2 - 17x - 9 = 0$

24. $4x^2 - 11x - 3 = 0$

25. $10x^2 - x - 2 = 0$

26. $2x^2 + 11x + 12 = 0$

27. $10x^2 - 27x + 5 = 0$

28. $3x^2 - 17x - 28 = 0$

29. $2x^2 - x - 28 = 0$

Using Factoring to Solve Quadratic Equations**Don't forget:**

See Chapter 6 for more information on factoring.

- 1) First arrange the terms in the quadratic equation so that you have **zero on one side**.
- 2) Then **factor** the nonzero expression (if possible).
- 3) Once done, you can use the **zero property** to find the solutions.

Example 4

Solve $x^2 - 6x - 7 = 0$.

Solution

The right-hand side of the equation is already zero, so you can just factor the left-hand side:

$$x^2 - 6x - 7 = (x + 1)(x - 7)$$

So $(x + 1)(x - 7) = 0$.

Using the zero property, either $x + 1 = 0$ or $x - 7 = 0$.

So either $x = -1$ or $x = 7$.

Example 5

Solve $x^2 + 2x - 11 = -3$.

Solution

This time, you have to arrange the equation so you have zero on one side. By adding 3 to both sides, the right-hand side becomes 0.

$$\begin{aligned}x^2 + 2x - 11 &= -3 \\ \Rightarrow x^2 + 2x - 8 &= 0\end{aligned}$$

Now you can factor the left-hand side:

$$x^2 + 2x - 8 = (x + 4)(x - 2) = 0$$

Since you have two expressions multiplied to give zero, you can use the zero property. That is, either $x + 4 = 0$ or $x - 2 = 0$.

So either $x = -4$ or $x = 2$.

Example 6

Solve $3x^2 + 168 = 45x$.

Solution

Once again, the first thing to do is get zero on one side:

$$3x^2 + 168 = 45x \Rightarrow 3x^2 - 45x + 168 = 0$$

The left-hand side can be factored, which means you can rewrite this as:

$$3(x^2 - 15x + 56) = 0, \text{ or } 3(x - 7)(x - 8) = 0$$

So using the zero property, either $x - 7 = 0$ or $x - 8 = 0$.

So either $x = 7$ or $x = 8$.

 **Guided Practice**

Solve each of these equations.

30. $x^2 - 2x - 15 = 0$

32. $k^2 + 10k + 24 = 0$

34. $8k^2 - 14k = 49$

36. $6y^2 + 28y + 20 = 5 - 6y^2$

38. $4(x^2 - 5x) = -25$

40. $x(x + 4) + 9 = 5$

42. $6x(3x - 4) - 7 = -15$

44. $7x(7x + 2) + 4x^3 + 3 = 2(2x^3 + 1)$

45. $(2x + 9)^2(x + 3)(x + 1)^{-1}(x + 3)^{-1}(x + 1) = 0$

46. $2x(3x + 3) + 4(x + 1) = 1 + 2x + 2x^2$

31. $x^2 - 7x - 18 = 0$

33. $4m^2 + 4m - 15 = 0$

35. $15k^2 + 28k = -5$

37. $4x^2 + 6x + 1 = 3x^2 + 8x$

39. $3x(3x + 4) + 8 = 4$

41. $x(x - 5) + 3 = -3$

43. $3 = 2 - 12x(3x - 1)$

✓ Independent Practice

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where a , b , and c are numbers. Identify a , b , and c in the quadratic equations below.

1. $4x^2 + 20x + 9 = 0$

2. $x^2 - 9x + 8 = 0$

3. $2x^2 + 5x = 35 + 14x$

4. $x(2x + 3) = 5(2x + 3)$

5. $y(2y + 7) = 9(2y + 7)$

6. $(x + 2)(x - 2) = 3x$

Use the zero product property to solve these equations.

7. $(2y + 9)(2y - 3) = 0$

8. $(2a + 5)(a - 11) = 0$

9. $(y - 3)^2(y - 5)(y - 3)^{-1} = 0$

10. $(y - 4)^3(2y - 9)^2(y - 4)^{-3}(y - 7)(2y - 9)^{-1} = 0$

Solve the following equations.

11. $4x^2 + 20x + 9 = 0$

12. $x^2 - 9x + 8 = 0$

13. $2x^2 + 5x = 35 + 14x$

14. $x(2x + 3) = 5(2x + 3)$

15. $y(2y + 7) = 9(2y + 7)$

16. $(x + 2)(x - 2) = 3x$

17. $2x(2x - 5) = 3(2x - 5)$

18. $2x(3x - 1) + 7 = 7(2 - 3x)$

19. The product of two consecutive positive numbers is 30. Find the numbers.

20. The product of two consecutive positive odd numbers is 35. Find the numbers.

21. The area of a rectangle is 35 ft^2 . If the width of the rectangle is $x \text{ ft}$ and the length is $(3x + 16) \text{ ft}$, find the value of x .

22. The area of a rectangle is 75 cm^2 . If the length of the rectangle is $(4x + 25) \text{ cm}$ and the width is $2x \text{ cm}$, find the dimensions of the rectangle.

23. Eylora has x pet goldfish and Leo has $(4x - 25)$. If the product of the numbers of Eylora's and Leo's goldfish is 21, how many goldfish does Leo have?

24. Scott fixed x computers and Meimei fixed $(5x - 7)$ computers. If the product of the number each fixed is 6, who fixed more computers?

Round Up

Don't forget that you need to *rearrange* the equation until you've got a *zero on one side* before you can factor a quadratic. Not all quadratics can be factored like this, as you'll see in the next Topic.

Quadratic Equations — Taking Square Roots

California Standards:

11.0: Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.

14.0: Students solve a quadratic equation by factoring or completing the square.

What it means for you:

You'll solve quadratic equations by taking square roots.

Key words:

- quadratic
- factor
- square root
- zero property

Some quadratics can be solved by taking square roots.

But to use this method properly, you have to remember something about squares and square roots.

The Square Root Method

If you square the numbers m and $-m$, you get the same answer, since $m^2 = (-m)^2 (= p, \text{ say})$.

If you take the **square root** of p , there are **two possible answers**, m or $-m$.

In other words, if $m^2 = p$, then $m = \pm\sqrt{p}$.

Example 1 uses the above property to find the **two possible solutions** of a quadratic equation.

Example 1

Solve the equation $x^2 = 25$.

Solution

Take the square root of both sides to get $x = \pm 5$.

In Example 1, you need only put the “ \pm ” on one side of the equation. Here's why:

Take the square root of both sides of the original equation to get $\sqrt{x^2} = \sqrt{25}$.

But when you take square roots, you have to allow for both sides to be either positive or negative. So actually, there are four possibilities here:

$$x = 5, -x = 5, x = -5, \text{ or } -x = -5$$

However, $x = 5$ and $-x = -5$ are the same, as are $-x = 5$ and $x = -5$.

So in fact it's enough to put the “ \pm ” sign on just **one side of the equation**.

You Can Also Solve this Equation by Factoring

You could also solve the above equation using the method of factoring from the previous pages.

But to use the factoring method, you have to have an equation of the form: “**something = 0**” — then you can use the **zero property**.

Example 2

Solve the equation $x^2 = 25$ by factoring.

Solution

$$\begin{aligned}x^2 &= 25 \\x^2 - 25 &= 0 \\(x + 5)(x - 5) &= 0\end{aligned}$$

So using the zero property, either $x + 5 = 0$ or $x - 5 = 0$.

So either $x = 5$ or $x = -5$. That is, $x = \pm 5$.

Check it out:

This is the same problem as in Example 1.

Guided Practice

Find the square roots of the expressions below. Show your work.

- | | | |
|-------------------|---------------------|---------------------|
| 1. 49 | 2. 64 | 3. 256 |
| 4. 128 | 5. $4t^2$ | 6. $16t^4$ |
| 7. $x^2 + 4x + 4$ | 8. $a^2 + 16a + 64$ | 9. $4y^2 + 12y + 9$ |

Solve the equations below by finding the square root.

- | | | |
|-----------------|----------------|------------------|
| 10. $x^2 = 4$ | 11. $x^2 = 9$ | 12. $2x^2 = 32$ |
| 13. $3a^2 = 75$ | 14. $a^2 = 81$ | 15. $5x^2 = 180$ |

16. Use the zero product property and factoring to verify your answers to Exercises 10–15 above.

More Square Root Examples

Example 3

Find the solution set of $3x^2 - 7 = 101$.

Solution

Here, you can get x^2 on its own on one side of the equation, with no x 's on the other side. This allows you to take the square root, as above.

$$\begin{aligned}3x^2 - 7 &= 101 \\3x^2 &= 108 \\x^2 &= 36 \\x &= \pm\sqrt{36}\end{aligned}$$

That is, $x = \pm 6$ — or $x \in \{6, -6\}$

Check it out:

$x = \pm 6$ and $x \in \{6, -6\}$ mean the same thing.

Example 4Solve $(x - 7)^2 = 64$.**Solution**

This time, you can't get x^2 on its own (with no x 's on the other side), but you already have $(x - 7)^2$ alone, which is just as good.

$$(x - 7)^2 = 64$$

$$x - 7 = \pm\sqrt{64}$$

$$x - 7 = \pm 8$$

$$x = 7 \pm 8$$

So $x = 15$ or $x = -1$.

Check it out:

You can't get x^2 on its own because if you multiply out these parentheses, you get $x^2 - 14x + 49$. So if you put x^2 on one side, you'll have $14x$ on the other side, which is no good.

Check Your Answers by Using the Original Equation

You can always **check** your answers by substituting your solutions into the original equation to see if you get a true statement.

This example shows how you'd check the solution reached in Example 4.

Example 5

Show that $x = 15$ and $x = -1$ are solutions of the equation $(x - 7)^2 = 64$.

Solution

Do this by substituting $x = 15$ and $x = -1$ into the equation $(x - 7)^2 = 64$, and seeing if you get true statements.

Put $x = 15$:

$$(15 - 7)^2 = 64$$

$$8^2 = 64$$

$$64 = 64$$

Put $x = -1$:

$$(-1 - 7)^2 = 64$$

$$(-8)^2 = 64$$

$$64 = 64$$



These are both true statements, so $x = 15$ and $x = -1$ are both solutions of the equation.

Guided Practice

Find the square roots of the expressions below.

17. $c^2 + 6c + 9$

18. $x^2 + 14x + 49$

19. $x^2 - 6x + 9$

20. $9x^2 - 24x + 16$

21. $9x^2 + 30x + 25$

22. $25 - 30k + 9k^2$

23. $49 + 28y + 4y^2$

24. $4x^2 + 4bx + b^2$

25. $k^2 - 8kx + 16x^2$

Solve the following equations by using the square root method.

26. $k^2 = 1$

27. $x^2 = 49$

28. $p^2 = 125$

29. $m^2 = 432$

30. $(x + 3)^2 = 81$

31. $(x - 5)^2 = 121$

32. $(k - 6)^2 = 72$

33. $(v + 7)^2 = 147$

34. $x^2 + 4x + 4 = 36$

A Couple More Examples

Example 6

Solve $4x^2 - 20x + 25 = 9$ by taking square roots of both sides.

Solution

$$4x^2 - 20x + 25 = 9$$

$$\sqrt{4x^2 - 20x + 25} = \sqrt{9}$$

$$\sqrt{(2x - 5)^2} = \sqrt{9} \quad \text{Since } (2x - 5)^2 = 4x^2 - 20x + 25$$

$$2x - 5 = \pm 3$$

So $2x = 8$ or $2x = 2$ — which means that $x = 4$ or $x = 1$.

Example 7

Solve $y^2 = 75$, giving your answer in its simplest form.

Solution

$$y^2 = 75$$

$$y = \pm\sqrt{75}$$

$$y = \pm\sqrt{25 \cdot 3}$$

$$y = \pm\sqrt{25} \cdot \sqrt{3}$$

$$y = \pm 5\sqrt{3}$$

Don't forget:

See Topics 1.3.2 and 1.3.3 for more information on expressions involving radicals.

Independent Practice

Find the square roots of the expressions below.

1. $25k^2 - 60kr + 36r^2$ 2. $\frac{4p^2 - 20p + 25}{4p^2 + 20p + 25}$ 3. $\frac{k^2 - 8k + 16}{4m^2 + 12m + 9}$

Solve the following equations by using the square root method.

4. $4x^2 - 4x + 1 = 16$ 5. $9x^2 + 12x + 4 = 169$ 6. $k^2 - 14k + 49 = \frac{9}{16}$

7. $4x^2 - 12x + 9 = 16$ 8. $9x^2 - 6x + 1 = 4$

9. The sides of a square are each $(2x - 16)$ cm long. Find the value of x that would give a square with an area of 108 cm^2 .

10. The product of the number of CDs that Donna and Keisha have is $16a^2 + 56a + 49$. If both have the same number of CDs, find how many CDs Donna has, in terms of a .

Round Up

Don't forget that square roots result in *two possible solutions*. Also, no matter how you've solved a quadratic, it's a good idea to *check* your solutions by substituting them back into the original equation.

Section 7.2 Completing the Square

California Standard:

14.0: Students solve a quadratic equation by factoring or completing the square.

What it means for you:

You'll form perfect square trinomials by adding numbers to binomial expressions.

Key words:

- completing the square
- perfect square trinomial
- binomial

Check it out:

The trinomial (with 3 terms) can be written as the square of a binomial (a binomial has 2 terms).

“Completing the square” is another method for solving quadratic equations — but before you solve any equations, you need to know how completing the square actually works.

Writing Perfect Square Trinomials as Perfect Squares

An expression such as $(x + 1)^2$ is called a **perfect square** — because it's *(something)*².

In a similar way, an expression such as $x^2 + 2x + 1$ is called a **perfect square trinomial** (“trinomial” because it has 3 terms).

This is because it can be written as a perfect square: $x^2 + 2x + 1 = (x + 1)^2$

Any trinomial of the form $x^2 + 2dx + d^2$ is a perfect square trinomial, since it can be written as the square of a binomial: $x^2 + 2dx + d^2 = (x + d)^2$

Converting Binomials to Perfect Squares

The binomial expression $x^2 + 4x$ is not a perfect square — it can't be written as the square of a binomial.

However, it can be turned into a perfect square trinomial if you add a constant (a number) to the expression.

Example 1

Convert $x^2 + 4x$ to a perfect square trinomial.

Solution

To do this you have to **add** a number to the original expression.

First look at the form of perfect square trinomials, and compare the **coefficient of x** with the **constant term** (the number not followed by x or x^2):

$$x^2 + 2dx + d^2 = (x + d)^2$$

The coefficient of x is $2d$, while the constant term is d^2 .

So the constant term is the **square of half of the coefficient of x** .

To convert $x^2 + 4x$ to a perfect square trinomial, add the square of half of 4 — that is, add $2^2 = 4$, to give $x^2 + 4x + 4 = (x + 2)^2$

✓ Guided Practice

By adding a constant, convert each of these binomials into a perfect square trinomial.

1. $x^2 + 14x$

2. $x^2 - 12x$

3. $x^2 + 2x$

4. $x^2 - 8x$

5. $y^2 + 20y$

6. $p^2 - 16p$

Completing the Square for $x^2 + bx$

Don't forget:

b is the coefficient of x .

Completing the square for $x^2 + bx$

To convert $x^2 + bx$ into a perfect square trinomial, **add** $\left(\frac{b}{2}\right)^2$.

The resulting trinomial is $\left(x + \frac{b}{2}\right)^2$.

Check it out:

The solution is equivalent to $(x + 4)^2$.

Example 2

Form a perfect square trinomial from $x^2 + 8x$.

Solution

Here, $b = 8$, so to complete the square you add $\left(\frac{8}{2}\right)^2$.

This gives you $x^2 + 8x + 16$.

Check it out:

In Example 3, $y^2 - 12y + 36$ could also be written as $(y - 6)^2$.

Example 3

What must be added to $y^2 - 12y$ to make it a perfect square trinomial?

Solution

This time, $b = -12$.

To complete the square you add $\left(\frac{-12}{2}\right)^2 = (-6)^2 = 36$.

So **36 must be added** (giving $y^2 - 12y + 36$).

Example 4

Suppose $x^2 - 10x + c$ is a perfect square trinomial, and is equal to $(x + k)^2$. What are the values of c and k ?

Solution

Here the coefficient of x is -10 . So to form a perfect square trinomial, the constant term has to be the **square of half of -10** , so $c = (-5)^2 = 25$.

Therefore $x^2 - 10x + c = x^2 - 10x + 25 = (x + k)^2$.

Now multiply out the parentheses of $(x + k)^2$.

$$(x + k)^2 = x^2 + 2kx + k^2$$

This has to equal $x^2 - 10x + 25$, which gives $x^2 - 10x + 25 = x^2 + 2kx + k^2$.

Equate the coefficients of x : the coefficient of x on the left-hand side is -10 , while on the right-hand side it is $2k$.

So $-10 = 2k$, or $k = -5$.

Comparing the constant terms in a similar way, you find that $25 = k^2$, which is also satisfied by $k = -5$.

Don't forget:

Remember the negative sign in front of the 10 coefficient.

Guided Practice

Find the value of k that will make each expression below a perfect square trinomial.

7. $x^2 - 7x + k$

8. $q^2 + 5q + k$

9. $x^2 + 6mx + k$

10. $d^2 - 2md + k$

Form a perfect square trinomial from the following expressions by adding a suitable term.

11. $x^2 + 10x$

12. $x^2 - 16x$

13. $y^2 + 2y$

14. $x^2 + bx$

15. $y^2 - 18y$

16. $a^2 - 2a$

17. $y^2 + 12y$

18. $y^2 + 36y$

19. $6x + 9$

20. $1 - 8x$

21. $25 - 20y$

22. $4y^2 + 4yb$

23. $4a^2 + 12ab$

24. $9a^2 + 16b^2$

The quadratics below are perfect square trinomials.

Find the value of c and k to make each statement true.

25. $x^2 - 6x + c = (x + k)^2$

26. $x^2 + 16x + c = (x + k)^2$

27. $4x^2 + 12x + c = (2x + k)^2$

28. $9x^2 + 30x + c = (3x + k)^2$

29. $4a^2 - 4ab + cb^2 = (2a + kb)^2$

30. $9a^2 - 12ab + cb^2 = (3a + kb)^2$

If the Coefficient of x^2 isn't 1, Add a Number

With an expression of the form $ax^2 + bx$, you can add a number to make an expression of the form $a(x + k)^2$.

Example 5

If $3x^2 - 12x + m$ is equal to $3(x + d)^2$, what is m ?

Solution

Multiply out the parentheses of $3(x + d)^2$ to get:

$$3(x + d)^2 = 3x^2 + 6dx + 3d^2$$

$$\text{So } 3x^2 - 12x + m = 3x^2 + 6dx + 3d^2$$

Equate the coefficients of x , and the constants, to get:

$$-12 = 6d \text{ and } m = 3d^2$$

The first equation tells you that $d = -2$.

And the second tells you that $m = 3(-2)^2$, or $m = 12$.

$$\text{So } 3x^2 - 12x + 12 = 3(x - 2)^2.$$

Completing the square for $ax^2 + bx$

The expression $ax^2 + bx$ can be changed to a trinomial of the form $a(x + k)^2$.

To do this, add $\frac{1}{a}\left(\frac{b}{2}\right)^2 = \frac{b^2}{4a}$.

The resulting trinomial is $a\left(x + \frac{b}{2a}\right)^2$.

Example 6

Convert $2x^2 + 10x$ to a perfect square trinomial.

Solution

Here, $a = 2$ and $b = 10$, so you add $\frac{1}{2}\left(\frac{10}{2}\right)^2 = \frac{1}{2}\left(\frac{100}{4}\right) = \frac{25}{2}$

This gives you $2x^2 + 10x + \frac{25}{2}$.

Check it out:

$2x^2 + 10x + \frac{25}{2}$ can also be written as $2\left(x + \frac{5}{2}\right)^2$.

✓ Guided Practice

The quadratics below are of the form $a(x + d)^2$.

Find the value of m and d in each equation.

- | | |
|-------------------------------------|-------------------------------------|
| 31. $5x^2 + 10x + m = 5(x + d)^2$ | 32. $4x^2 - 24x + m = 4(x + d)^2$ |
| 33. $2x^2 - 28x + m = 2(x + d)^2$ | 34. $3x^2 - 30x + m = 3(x + d)^2$ |
| 35. $4x^2 + 32x + m = 4(x + d)^2$ | 36. $20x^2 + 60x + m = 5(2x + d)^2$ |
| 37. $20x^2 - 20x + m = 5(2x + d)^2$ | 38. $27x^2 + 18x + m = 3(3x + d)^2$ |
| 39. $27x^2 + 36x + m = 3(3x + d)^2$ | 40. $16x^2 - 80x + m = 4(2x + d)^2$ |

Add a term to convert each of the following into an expression of the form $a(x + k)^2$.

- | | |
|------------------|------------------|
| 41. $2x^2 - 12x$ | 42. $3a^2 + 12a$ |
| 43. $6y^2 - 60y$ | 44. $4x^2 - 48x$ |
| 45. $5x^2 + 245$ | 46. $8x^2 + 2$ |
| 47. $36x + 12$ | 48. $120x + 100$ |

✓ Independent Practice

Find the value of c that will make each expression below a perfect square trinomial.

- | | |
|---------------------|---------------------|
| 1. $x^2 + 9x + c$ | 2. $x^2 - 11x + c$ |
| 3. $x^2 + 12xy + c$ | 4. $x^2 - 10xy + c$ |

Complete the square for each quadratic expression below.

- | | |
|-----------------|-----------------|
| 5. $x^2 - 6x$ | 6. $a^2 - 14a$ |
| 7. $b^2 - 10b$ | 8. $x^2 + 8xy$ |
| 9. $c^2 - 12bc$ | 10. $x^2 + 4xy$ |

Find the value of m and d in each of the following.

- | | |
|-----------------------------------|-----------------------------------|
| 11. $5x^2 - 40x + m = 5(x + d)^2$ | 12. $2x^2 + 20x + m = 2(x + d)^2$ |
| 13. $3x^2 - 6x + m = 3(x + d)^2$ | 14. $3x^2 - 30x + m = 3(x + d)^2$ |
| 15. $4x^2 + 24x + m = 4(x + d)^2$ | 16. $7x^2 - 28x + m = 7(x + d)^2$ |

Add a term to convert each of the following into an expression of the form $a(x + k)^2$.

- | | |
|-------------------|-------------------|
| 17. $3x^2 - 30x$ | 18. $2x^2 + 8x$ |
| 19. $18x^2 - 48x$ | 20. $5x^2 + 180$ |
| 21. $27x^2 + 12$ | 22. $36x^2 + 196$ |

23. The length of a rectangle is twice its width. If the area can be found by completing the square for $(18x^2 + 60x)$ ft², find the width of the rectangle.

Round Up

OK, so now you know how to add a number to a binomial to make a *perfect square trinomial*. In the next Topic you'll learn how to convert any quadratic expression into perfect square trinomial form — and then in Topic 7.2.3 you'll use this to *solve quadratic equations*.

More on Completing the Square

California Standards:

14.0: Students solve a quadratic equation by factoring or completing the square.

What it means for you:

You'll convert any quadratic expression into perfect square trinomial form.

Key words:

- completing the square
- perfect square trinomial
- binomial

In Topic 7.2.1, you converted binomial expressions like $ax^2 + bx$ to perfect square trinomials by adding a number.

In this Topic you'll take a more general quadratic expression like $ax^2 + bx + c$ and write this in the form $a(x + k)^2 + m$.

Writing $x^2 + bx + c$ in the form $(x + k)^2 + m$

In earlier Topics, you converted an expression of the form $x^2 + bx$ to a perfect square trinomial by adding $\left(\frac{b}{2}\right)^2$ to it.

Example 1

Convert $x^2 + 4x$ to a perfect square trinomial.

Solution

Here $b = 4$, so to convert this to a perfect square trinomial, you add

$$\left(\frac{4}{2}\right)^2 = 2^2 = 4.$$

So $x^2 + 4x + 4 = (x + 2)^2$. Here $x^2 + 4x + 4$ is a perfect square trinomial.

Another way to think about this is to say that your original expression was equal to an expression of the form $(x + k)^2 + m$.

Example 2

Express $x^2 + 4x$ in the form $(x + k)^2 + m$.

Solution

$$x^2 + 4x + 4 = (x + 2)^2.$$

$$\text{Therefore } x^2 + 4x = (x + 2)^2 - 4.$$

Check it out:

Example 2 is almost the same problem as in Example 1 — it's just asked in a different way.

✓ Guided Practice

Express each of the following in the form $(x + k)^2 + m$.

1. $x^2 + 6x$

2. $x^2 + 20x$

3. $x^2 + 12x$

4. $x^2 + 24x$

5. $x^2 - 22x$

6. $x^2 - 10x$

7. $x^2 + 18x$

8. $x^2 - 14x$

9. $x^2 - 16x$

More general quadratic expressions $x^2 + bx + c$ can also be written in the form $(x + k)^2 + m$.

Writing $x^2 + bx + c$ in the form $(x + k)^2 + m$

- 1) First take just the first two terms ($x^2 + bx$) and convert this to a perfect square trinomial by adding the **square of half of b** :

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

- 2) Rewrite this in the form $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

- 3) Add c to both sides: $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$

Check it out:

Compare this to $(x + k)^2 + m$, and you see that

$$k = \frac{b}{2} \text{ and } m = -\left(\frac{b}{2}\right)^2 + c.$$

Example 3

Express $x^2 + 4x + 1$ in the form $(x + k)^2 + m$.

Solution

- 1) Again, $b = 4$, so add $\left(\frac{4}{2}\right)^2 = 2^2 = 4$ to the first two terms of your expression to find a perfect square trinomial — that is, $x^2 + 4x + 4 = (x + 2)^2$.

So the first two terms of your original expression can be expressed:

$$x^2 + 4x = (x + 2)^2 - 4$$

- 2) Now add $c (= 1)$ to both sides of this equation to get:

$$x^2 + 4x + 1 = (x + 2)^2 - 3$$

Example 4

Write $x^2 - 6x + 3$ in the form $(x + k)^2 + m$.

Solution

- 1) Here, $b = -6$, so add $\left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$ to $x^2 - 6x$ for a perfect square trinomial — that is, $x^2 - 6x + 9 = (x - 3)^2$.

So the first two terms of the original quadratic can be expressed:

$$x^2 - 6x = (x - 3)^2 - 9$$

- 2) Add $c (= 3)$ to both sides of this equation to get:

$$x^2 - 6x + 3 = (x - 3)^2 - 6$$

✓ Guided Practice

Express the following in the form $(x + k)^2 + m$ [or $a(x + k)^2 + m$].

10. $x^2 + 4x + 8$

12. $x^2 + 8x + 5$

14. $x^2 + 3x + 5$

16. $x^2 + 4x + 5$

18. $x^2 - 20x + 20$

20. $4x^2 + 16x + 4$

11. $x^2 + 6x + 14$

13. $x^2 - 12x + 8$

15. $x^2 - 5x - 7$

17. $x^2 - 6x + 30$

19. $x^2 + 22x + 54$

21. $2x^2 + 20x + 25$

Writing $ax^2 + bx + c$ in the Form $a(x + k)^2 + m$

This is the most general case you can meet — writing $ax^2 + bx + c$ as an expression of the form $a(x + k)^2 + m$. The method's very similar to the one used in the previous examples.

Example 5

Write $2x^2 + 6x + 7$ in the form $a(x + k)^2 + m$.

Solution

1) Factor your expression by taking the coefficient of x^2 outside

$$\text{parentheses: } 2x^2 + 6x + 7 = 2\left(x^2 + 3x + \frac{7}{2}\right)$$

2) The first two terms in parentheses ($x^2 + 3x$) can be made into a perfect

square trinomial in the usual way — by adding $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

$$\text{This means that } x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2$$

$$\text{— that is, } x^2 + 3x = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}.$$

$$\text{So } x^2 + 3x + \frac{7}{2} = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{7}{2}$$

$$\text{or, combining the fractions, } x^2 + 3x + \frac{7}{2} = \left(x + \frac{3}{2}\right)^2 + \frac{5}{4}$$

3) Therefore the original expression ($2x^2 + 6x + 7$) can be written:

$$2\left[\left(x + \frac{3}{2}\right)^2 + \frac{5}{4}\right] = 2\left[x + \frac{3}{2}\right]^2 + \frac{5}{2}$$

Check it out:

Now you can deal with the expression in parentheses exactly as before.

Check it out:

This is exactly the same process as in Examples 3 and 4 on the previous page.

Example 6

Write $4x^2 - 4x + 3$ in the form $a(x + k)^2 + m$.

Solution

1) Factor the expression: $4x^2 - 4x + 3 = 4\left(x^2 - x + \frac{3}{4}\right)$

2) Convert the first two terms from inside the parentheses into a perfect square trinomial: $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$

So rewriting the above, you get: $x^2 - x = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$

Add $\frac{3}{4}$ to both sides: $x^2 - x + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{1}{2}$

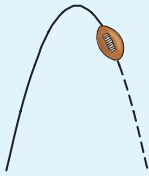
3) Finally, use your expression from 1) above:

$$4x^2 - 4x + 3 = 4\left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}\right]$$

$$= 4\left(x - \frac{1}{2}\right)^2 + 2$$

Check it out:

Completing the square can be used to find the highest point of an object's path as it flies through the air.



You'll see this in more detail in Sections 7.4 and 7.6.

Guided Practice

Express the following in the form $(x + k)^2 + m$ [or $a(x + k)^2 + m$].

22. $2x^2 - 9x + 4$

23. $2x^2 + 9x + 5$

24. $3x^2 - 12x + 14$

25. $-x^2 + 2x + 3$

26. $-x^2 - 3x + 4$

27. $-2x^2 + 6x + 10$

28. $-3x^2 - 9x + 21$

29. $-4x^2 + 120x - 80$

Independent Practice

Express the following in the form $(x + k)^2 + m$.

1. $x^2 + 4x$

2. $x^2 - 6x$

3. $x^2 + 10x$

4. $x^2 - 20x$

5. $x^2 - \frac{1}{4}x$

6. $x^2 + \frac{2}{3}x$

Express the following in the form of $a(x + k)^2 + m$

7. $-5x^2 - 90x - 150$

8. $2x^2 + 5x + 10$

9. $4x^2 - 3x + 12$

10. $5x^2 + 20x + 4$

11. $6x^2 + 48x + 16$

12. $3x^2 + 42x + 49$

13. A square has an area of $x^2 + 14x + k$. Find the value of k .

14. A circle has an area of $\pi x^2 + 18\pi x + k\pi$. Find the value of k .

Round Up

There's been a lot of build-up to actually solving quadratic equations using the completing the square method — but it's coming up next, in Topic 7.2.3.

Solving Equations by Completing the Square

California Standards:

14.0: Students solve a quadratic equation by factoring or completing the square.

What it means for you:

You'll use completing the square to solve quadratic equations.

Key words:

- completing the square
- perfect square trinomial
- binomial

Check it out:

The aim is to get a perfect square involving x on one side of the equation.

This is what the whole Section has been leading up to.

*The process of **completing the square** is a really useful method that can help solve quadratic equations.*

You can Solve Quadratics by Completing the Square

The best way to show this is with an example:

Example 1

Solve $x^2 - 10x + 21 = 0$ by completing the square.

Solution

To do this, you apply some of the techniques from the earlier parts of this chapter.

- 1) Take the constant (= 21) onto the right-hand side of the equation. Then convert what remains on the left-hand side ($x^2 - 10x$) to a **perfect square trinomial** by adding the square of half the coefficient of x (to both sides of the equation) — so add $(-5)^2 = 25$.

$$\begin{aligned}x^2 - 10x &= -21 \\x^2 - 10x + 25 &= -21 + 25 \\x^2 - 10x + 25 &= 4\end{aligned}$$

- 2) Now you can use the **square root method** to solve the equation.

$$\begin{aligned}x^2 - 10x + 25 &= 4 \\(x - 5)^2 &= 4 \\x - 5 &= \pm 2 \\x - 5 = 2 \quad \text{or} \quad x - 5 &= -2 \\x = 7 \quad \text{or} \quad x &= 3\end{aligned}$$

Example 2

Solve the following quadratic equation: $4x^2 - 9x + 2 = 0$

Solution

The best thing to do here is to divide the equation by 4 first.

Then you need to solve: $x^2 - \frac{9}{4}x + \frac{1}{2} = 0$

1) $x^2 - \frac{9}{4}x = -\frac{1}{2}$ **Move the constant to the other side**

$$x^2 - \frac{9}{4}x + \left(\frac{9}{8}\right)^2 = -\frac{1}{2} + \left(\frac{9}{8}\right)^2 = \frac{49}{64}$$
 Add a number to both sides to get a perfect square trinomial

$$\left(x - \frac{9}{8}\right)^2 = \frac{49}{64}$$
 Rewrite the perfect square trinomial as a square

So now you have to solve: $\left(x - \frac{9}{8}\right)^2 = \frac{49}{64}$

2) Use the square root method.

$$\left(x - \frac{9}{8}\right)^2 = \frac{49}{64}$$

$$x - \frac{9}{8} = \pm \frac{7}{8}$$

$$x - \frac{9}{8} = \frac{7}{8} \quad \text{or} \quad x - \frac{9}{8} = -\frac{7}{8}$$

$$x = \frac{16}{8} = 2 \quad \text{or} \quad x = \frac{2}{8} = \frac{1}{4}$$

Guided Practice

Solve the following equations by completing the square.

1. $x^2 - 6x + 5 = 0$

2. $x^2 + 4x - 5 = 0$

3. $x^2 - 10x - 200 = 0$

4. $x^2 - 12x - 64 = 0$

5. $x^2 + 14x + 33 = 0$

6. $x^2 - 8x + 12 = 0$

7. $x^2 - 2x - 8 = 0$

8. $x^2 - 16x + 39 = 0$

9. $x^2 + 18x - 19 = 0$

10. $x^2 - 20x - 44 = 0$

11. $x^2 + 22x + 21 = 0$

12. $2x^2 - 12x + 10 = 0$

13. $3a^2 + 12a - 15 = 0$

14. $2y^2 + 20y + 18 = 0$

Example 3

$$\text{Solve } 5x^2 - 2x - 3 = 3x^2 - 6x + 13.$$

Solution

Rearrange the equation first so that all like terms are combined:

$$\begin{aligned} 5x^2 - 2x - 3 &= 3x^2 - 6x + 13 \\ 2x^2 + 4x - 16 &= 0 \end{aligned}$$

Now divide through by 2 so that the coefficient of x^2 is 1.

$$x^2 + 2x - 8 = 0$$

Take the constant to the other side — then convert what's left to a perfect square.

$$\begin{aligned} x^2 + 2x &= 8 \\ x^2 + 2x + 1 &= 8 + 1 \\ (x + 1)^2 &= 9 \end{aligned}$$

Use the completed square to solve the original equation.

$$\begin{aligned} (x + 1)^2 &= 9 \\ x + 1 &= \pm 3 \\ \text{So } x + 1 &= 3 \quad \text{or} \quad x + 1 = -3 \\ \mathbf{x = 2} \quad \text{or} \quad \mathbf{x = -4} \end{aligned}$$

Check it out:

You could complete the square at this stage, but it's easier if the coefficient of x^2 is 1.

Example 4

$$\text{Find } t \text{ if } 24t - 3t^2 - 48 = -9.$$

Solution

$$\begin{aligned} 24t - 3t^2 - 48 &= -9 \\ 3t^2 - 24t + 39 &= 0 \\ t^2 - 8t + 13 &= 0 \\ t^2 - 8t &= -13 \end{aligned}$$

$$\begin{aligned} t^2 - 8t + \left(\frac{8}{2}\right)^2 &= -13 + \left(\frac{8}{2}\right)^2 \\ (t - 4)^2 &= -13 + 16 \\ (t - 4)^2 &= 3 \\ t - 4 &= \pm\sqrt{3} \end{aligned}$$

Add a number to both sides to get a perfect square trinomial

$$\text{So } \mathbf{t = 4 + \sqrt{3}} \quad \text{or} \quad \mathbf{t = 4 - \sqrt{3}}$$

✓ Guided Practice

Solve each of these quadratic equations using the method of completing the square:

15. $x^2 - 2x - 15 = 0$

16. $x^2 + 2x - 24 = 0$

17. $x^2 + 14x + 45 = 0$

18. $x^2 - 15x + 56 = 0$

19. $5x^2 + 12x + 38 = 2x^2 + 36x + 2$

20. $-150 + 2b^2 = 20b$

21. $5a(a + 2) - 120 = 0$

22. $5y(y + 2) - 13 = 2(y^2 - y + 1)$

23. $5x^2 + 44x - 49 = 2x(1 - x)$

24. $4x(x - 11) - 4 = 4(x - 12)$

25. $6a(a + 3) = 6(a - 1) + 216$

26. $5y(y - 6) + 51 = 131$

27. $7b^2 + 14b - 32 = 5b^2 + 26b$

28. $3(x^2 - 5) - 9 = 111 - 12x$

29. $8(x + 6)^2 - 128 = 6(x + 6)^2$

30. $4(x + 5)^2 - 200 = 2(x + 5)^2$

✓ Independent Practice

Solve by completing the square.

1. $x^2 + 2x - 3 = 0$

2. $y^2 - 4y - 12 = 0$

3. $a^2 + 8a - 9 = 0$

4. $b^2 + 10b - 24 = 0$

5. $x^2 - 12x + 20 = 0$

6. $3x^2 - 42x + 39 = 0$

7. $2a^2 + 12a + 10 = 0$

8. $5b^2 + 20b - 105 = 0$

9. $7d^2 - 14d - 21 = 0$

10. $2x^2 + 48x - 50 = 0$

11. $3c^2 + 66c + 63 = 0$

12. $3y(y - 8) + 2 = 254$

13. $6c(c - 4) = 2(18c + 168)$

14. $5(a^2 + 14) = 5 - 70a$

15. $y^2 - 2py - 15p^2 = 0$

16. $x^2 - 3kx + 2k^2 = 0$

17. An expression for the area of a rectangle is $(2x^2 + 4x)$ cm² and is equal to 6 cm². Find the value of x .

18. An expression for the area of a circle is $(4\pi y^2 - 16\pi y)$ ft² and is equal to 20π ft². Find the value of y .

Round Up

Make sure you understand all the examples — then once you understand the method, get plenty of practice. Completing the square means first forming a *perfect square trinomial*, and then using the new form to solve the original equation.

California Standards:

19.0: Students know the quadratic formula and are familiar with its proof by completing the square.
20.0: Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.

What it means for you:

You'll use the quadratic formula to solve quadratic equations — and you'll derive the quadratic formula itself.

Key words:

- quadratic formula
- completing the square

You can also use the *quadratic formula* to solve quadratic equations. It works every time.

Quadratic Equations can be in Any Variable

The standard form for a quadratic equation is

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

Any quadratic equation can be written in this form by, if necessary, rearranging it so that **zero** is on one side.

A lot of the quadratic equations you will see may contain a variable other than x — but they are still quadratic equations like the one above, and can be solved in the same way.

Example 1

Find the solutions of the following quadratic equations:

a) $x^2 - 4x + 3 = 0$ b) $y^2 - 4y + 3 = 0$

Solution

a) Here, $a = 1$, $b = -4$, and $c = 3$.

This equation factors to give $(x - 3)(x - 1) = 0$

So using the zero property:

$(x - 3) = 0$ or $(x - 1) = 0$, or **$x = 3$** or **$x = 1$**

b) Here, the variable is y rather than x — but that does not affect the solutions.

Again $a = 1$, $b = -4$, and $c = 3$, so it is the same equation as in a), and will have the same solutions:

$y = 3$ or **$y = 1$**

You can see that the two quadratic equations are really the same — only the variables have changed.

The Quadratic Formula Solves Any Quadratic Equation

The solutions to the quadratic equation $ax^2 + bx + c = 0$ are given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Check it out:

The quadratic formula gives you the solutions of any quadratic equation in terms of a , b , and c . It's sometimes called the general solution of the quadratic equation $ax^2 + bx + c = 0$.

Don't forget:

See Section 7.2 for a reminder about completing the square.

You can derive the quadratic formula by **completing the square**:

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Dividing the equation by a

$$\left[x + \frac{b}{2a}\right]^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Completing the square

$$\left[x + \frac{b}{2a}\right]^2 = \frac{-4ac + b^2}{4a^2}$$

$$\left[x + \frac{b}{2a}\right]^2 = \frac{b^2 - 4ac}{4a^2}$$

Rearranging

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Include positive and negative roots

$$\text{So } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Examples Using the Quadratic Formula

Example 2

Solve $x^2 - 5x - 14 = 0$ using the quadratic formula.

Solution

Start by writing down the values of a , b , and c :

$$a = 1, b = -5, \text{ and } c = -14$$

Now very carefully substitute these values into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 1 \cdot (-14)}}{2 \cdot 1}$$

$$x = \frac{5 \pm \sqrt{25 + 56}}{2} = \frac{5 \pm \sqrt{81}}{2} = \frac{5 \pm 9}{2}$$

$$\text{So } x = \frac{5+9}{2} = 7 \text{ or } x = \frac{5-9}{2} = -2$$

Don't forget:

Be very careful if there are minus signs.

Check it out:

You can check these answers by substituting them into the original equation:

$$7^2 - (5 \times 7) - 14 = 0$$

$$(-2)^2 - (5 \times (-2)) - 14 = 0.$$

So both solutions are correct.

It's really important to practice with the quadratic formula — you have to be able to use it correctly.

Example 3

Solve $2x^2 - 3x - 2 = 0$ using the quadratic formula.

Solution

$$a = 2, b = -3, \text{ and } c = -2$$

Putting these values into the quadratic formula...

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot (-2)}}{2 \cdot 2}$$

$$x = \frac{3 \pm \sqrt{9 + 16}}{4} = \frac{3 \pm \sqrt{25}}{4} = \frac{3 \pm 5}{4}$$

$$\text{So } x = \frac{3+5}{4} = 2 \text{ or } x = \frac{3-5}{4} = -\frac{1}{2}$$

Example 4

Solve $2x^2 - 11x + 13 = 0$.

Solution

$$a = 2, b = -11, \text{ and } c = 13$$

Putting these values into the quadratic formula...

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4 \cdot 2 \cdot (13)}}{2 \cdot 2}$$

$$x = \frac{11 \pm \sqrt{121 - 104}}{4} = \frac{11 \pm \sqrt{17}}{4}$$

$$\text{So } x = \frac{11 + \sqrt{17}}{4} \text{ or } x = \frac{11 - \sqrt{17}}{4}$$

Check it out:

Leave these answers as radical expressions unless you're told otherwise.

✓ Guided Practice

Use the quadratic formula to solve each of the following equations.

1. $x^2 - 2x - 143 = 0$

2. $2x^2 + 3x - 1 = 0$

3. $x^2 + 2x - 1 = 0$

4. $x^2 + 3x + 1 = 0$

5. $2x^2 - 5x + 2 = 0$

6. $3x^2 - 2x - 3 = 0$

7. $2x^2 - 7x - 3 = 0$

8. $6x^2 - x - 1 = 0$

9. $18x^2 + 3x - 1 = 0$

10. $4x^2 - 5x + 1 = 0$

11. The equation $2x^2 - 7x - 4 = 0$ factors to $(2x + 1)(x - 4) = 0$.

Using the zero product property we can find that $x = -\frac{1}{2}$ or $x = 4$.

Verify this using the quadratic formula.

12. The height of a triangle is 4 ft more than 4 times its base length. If the triangle's area is $\frac{5}{2}$ ft², find the length of its base.

✓ Independent Practice

Use the quadratic formula to solve each of the following equations.

1. $5x^2 - 11x + 2 = 0$

2. $2x^2 + 7x + 3 = 0$

3. $7x^2 + 6x - 1 = 0$

4. $x^2 - 7x + 5 = 0$

5. $10x^2 + 7x + 1 = 0$

6. $3y^2 - 8y - 3 = 0$

7. $5x^2 - 2x - 3 = 0$

8. $4x^2 + 3x - 5 = 0$

9. $4t^2 + 7t - 2 = 0$

10. $6m^2 + m - 1 = 0$

11. $2x^2 - x = 1$

12. $3x^2 - 5x = 2$

13. $2x^2 + 7x = 4$

14. $4x^2 + 17x = 15$

15. $4x^2 - 13x + 3 = 0$

16. $4x^2 - 1 = 0$

17. $25x^2 - 9 = 0$

18. $4x^2 + 15x = 4$

19. $10x^2 + 1 = 7x$

20. $16x^2 + 3 = 26x$

Solve these equations by factoring and using the zero product property, then verify the solutions by solving them with the quadratic formula.

21. $x^2 + 4x + 4 = 0$

22. $4y^2 - 9 = 0$

23. $x^2 - x - 12 = 0$

24. $2x^2 - 3x - 9 = 0$

25. $6x^2 + 29x = 5$

26. $7x^2 + 41x = 6$

27. The length of a rectangle is 20 cm more than 4 times its width. If the rectangle has an area of 75 cm², find its dimensions.

28. The equation $h = -14t^2 + 12t + 2$ gives the height of a tennis ball t seconds after being hit. How long will the ball take before it hits the ground?

Round Up

The quadratic formula looks quite complicated, but don't let that put you off. If you work through the derivation of the formula on p356 then you should see exactly why it contains all the elements it does.

Applications of Quadratics

California Standards:

20.0: Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.

What it means for you:

You'll model real-life problems using quadratic equations and then solve them using the quadratic formula.

Key words:

- quadratic formula
- completing the square

Check it out:

The mathematical model in Example 1 is the statement that the numbers should be represented by x and $(x + 9)$. It's very important to be clear here to avoid confusion later.

Check it out:

In fact, this equation factors. But if you didn't spot that, use the quadratic formula as in the example.

Check it out:

Careful — you haven't finished yet. Now you need to go back to your initial model and **interpret** the solutions of your quadratic.

Sometimes you will have to make a “mathematical model” first, and then *solve* the equations you get from it.

Then when you get your solutions, you have to *interpret* them.

Modeling Means Writing Your Own Equations

Here's an example of modeling a real-life situation as a quadratic equation:

Example 1

The difference between a pair of numbers is 9.
Find all such pairs of numbers that have a product of 220.

Solution

There are two different numbers in each pair — call the lower number x . Then the higher number is $x + 9$.

Write the information from the question in the form of an equation:

$$x(x + 9) = 220$$

This is a quadratic equation, so rearrange it to the form $ax^2 + bx + c = 0$.

$$x(x + 9) = 220$$

$$x^2 + 9x = 220$$

$$x^2 + 9x - 220 = 0$$

Write down a , b , and c :

$$a = 1, b = 9, \text{ and } c = -220$$

Now you can use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-9 \pm \sqrt{9^2 - 4 \cdot 1 \cdot (-220)}}{2 \cdot 1}$$

$$x = \frac{-9 \pm \sqrt{81 + 880}}{2} = \frac{-9 \pm \sqrt{961}}{2} = \frac{-9 \pm 31}{2}$$

$$\text{So } x = \frac{-40}{2} = -20 \text{ or } x = \frac{22}{2} = 11.$$

So there are two possible values for x (where x is the lower of the two numbers): $x = -20$ or $x = 11$

The higher of the two numbers is found by adding 9 to each of these values. So there are two possible pairs of numbers, and they are:

-20 and -11 and **11 and 20**

Example 2

Find the dimensions of the rectangle whose length is 7 inches more than twice its width, and whose area is 120 in^2 .

Solution

Let x = width in inches. Then the length is $2x + 7$ inches.

From the question: $x(2x + 7) = 120$

Rearrange this quadratic into standard form: $2x^2 + 7x - 120 = 0$

Now use the formula with $a = 2$, $b = 7$, and $c = -120$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 2 \cdot (-120)}}{2 \cdot 2}$$

$$x = \frac{-7 \pm \sqrt{49 + 960}}{4} = \frac{-7 \pm \sqrt{1009}}{4} = \frac{-7 \pm 31.76}{4}$$

So $x = 6.19$ or $x = -9.69$

But the width cannot be negative, so you can ignore $x = -9.69$.

So the width must be $x = 6.2 \text{ in.}$ (to 1 decimal place), and the length is then $2x + 7 = (2 \times 6.19) + 7 = 19.4 \text{ in.}$ (to 1 decimal place).

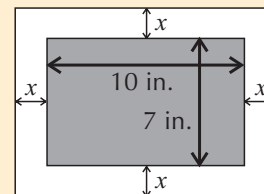
 **Guided Practice**

1. The difference between two numbers is 7. Find all possible pairs of such numbers if the product of the two numbers is 198.
2. Find the dimensions of a rectangular garden whose length is 10 meters more than three times its width, if the area is 77 m^2 .
3. Twice the square of a number is equal to eight times the number. Find the number.
4. The sum of the squares of two consecutive odd integers is 74. Find the numbers.
5. The sum of the squares of two consecutive even integers is 340. Find the possible numbers.
6. The length of a rectangular field is 10 meters less than four times its width. Find the dimensions if its area is 750 square meters.
7. When 15 and 19 are each increased by t , the product of the resulting numbers is 837. Find the value(s) of t .
8. A mother is three times as old as her daughter. Four years ago the product of their ages was 256. Find their current ages.

✓ Independent Practice

1. A man is five times older than his son. In three years' time, the product of their ages will be 380. Find their ages now.
2. Lorraine is 10 years older than Ahanu. In three years' time the product of their ages will be 600. Find Ahanu and Lorraine's ages now.

3. A picture of 10 inches by 7 inches is in a frame whose area (including the space for the picture) is 154 square inches. Find the dimensions of the frame if the gap between the edge of the picture and the frame is the same all the way around.



4. Jennifer has a picture of her boyfriend Zach measuring 10 inches by 8 inches. She frames the picture in a frame that has an area of 224 square inches (including the space for the picture). Find the dimensions of the picture frame if the gap between the edge of the picture and the frame is the same all the way around.
5. A wire of length 50 feet is bent to form a rectangular figure that has no overlap. If the area of the figure formed is 144 square feet, find the dimensions of the figure.
6. A piece of wire 22 yards long is bent to form a rectangular figure whose area is 28 square yards. Find the dimensions of the figure, given that there is no overlap in the wire.
7. Show that the sum of the solutions of $4x^2 - 4x - 3 = 0$ is equal to 1 ($= -\frac{b}{a}$).
8. Show that the product of the solutions of $x^2 - 7x + 10 = 0$ is equal to 10 ($= \frac{c}{a}$).
9. Using the quadratic formula, show that the sum of the solutions of the general quadratic equation $ax^2 + bx + c = 0$ is equal to $-\frac{b}{a}$, and that the product of the roots is $\frac{c}{a}$.

Round Up

Quadratic equations pop up a lot in Algebra I. If you know the quadratic formula then you'll always be able to solve them by just substituting the values into the formula.

Topic 7.4.1

Section 7.4

Graphs of Quadratic Functions

California Standards:

21.0: Students graph quadratic functions and know that their roots are the x -intercepts.

What it means for you:

You'll learn about the shape of various quadratic graphs.

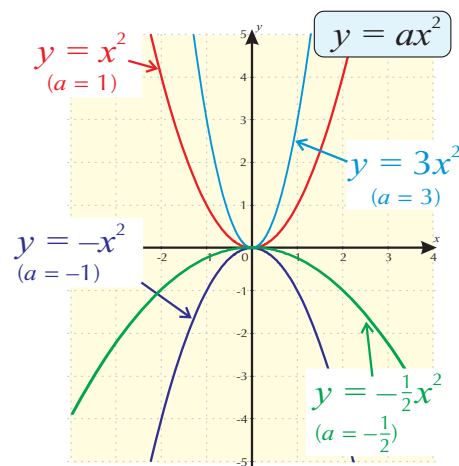
Key words:

- quadratic
- parabola
- concave
- vertex
- line of symmetry
- root

So far in this Chapter you've *solved quadratic equations in several different ways*. In this Section you'll see how the *graphs of quadratic functions can be plotted using the algebraic methods you've already seen*.

The Graphs of Quadratic Functions are Parabolas

If you plot the graph of any quadratic function, you get a curve called a **parabola**. The graphs of $y = ax^2$ (for various values of a) on the right show the basic shape of any quadratic graph.



- The parabola's either a u-shaped or n-shaped curve depending on the sign of a . The graph of $y = ax^2$ is **concave up** (u-shaped — it opens upwards) when $a > 0$, but **concave down** (n-shaped — it opens downwards) when $a < 0$.
- All quadratic graphs have one **vertex** (maximum or minimum point). For the curves shown above, the vertex is at the origin $(0, 0)$.
- All quadratic graphs have a vertical **line of symmetry**. For the graphs above, the line of symmetry is the y -axis.
- Notice that a bigger value of $|a|$ results in a steeper (narrower) parabola. For example, the graph of $y = 3x^2$ is steeper than the graph of $y = x^2$.

The basic shape of all quadratic graphs (that is, for any quadratic function $y = ax^2 + bx + c$) is very similar to the ones above.

They're all concave up or concave down depending on the sign of a (concave up if $a > 0$ and concave down if $a < 0$).

However, the graph can be **stretched** or **squashed**, and in a **different place** relative to the x - and y -axes, depending on the exact values of a , b , and c .

✓ Guided Practice

Match the equations with their graphs on the right.

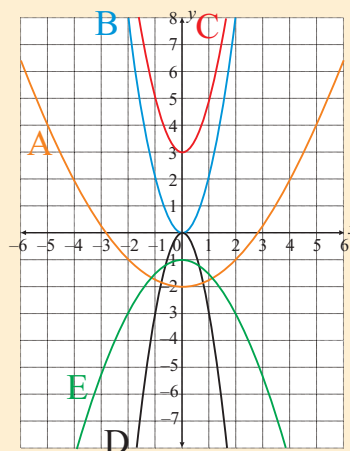
1. $y = -3x^2$

2. $y = \frac{1}{4}x^2 - 2$

3. $y = 2x^2 + 3$

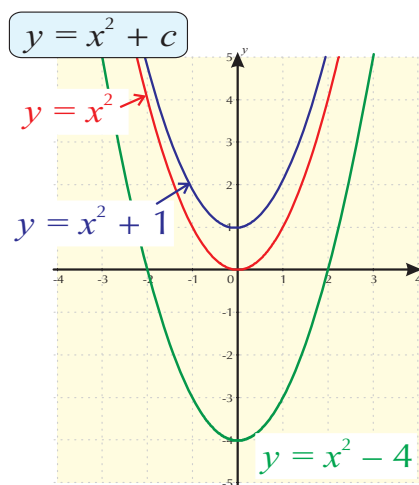
4. $y = -\frac{1}{2}x^2 - 1$

5. $y = 2x^2$



$y = ax^2 + c$ is Like $y = ax^2$ but Moved Up or Down by c

This diagram shows the graphs of $y = x^2 + c$, for three values of c :



- The top and bottom parabolas in the diagram are both the **same shape** as the graph of $y = x^2$. The only differences are:

- the graph of $y = x^2 + 1$ is 1 unit higher up the y -axis.
- the graph of $y = x^2 - 4$ is 4 units lower down the y -axis.

- The graph of $y = x^2 - 4$ crosses the x -axis when $y = 0$ — that is, when $x^2 - 4 = 0$ (or $x = \pm 2$).

In fact, the **x -intercepts** of any quadratic graph $y = ax^2 + bx + c$ are called the **roots** of the function, and they correspond to the **solutions** of the equation $ax^2 + bx + c = 0$.

- The graph of $y = x^2 + 1$ does not cross the x -axis at all. This is because $x^2 + 1 = 0$ does not have any real solutions.

So the graph of a quadratic function may cross the x -axis twice ($y = x^2 - 4$), may touch the x -axis in one place ($y = x^2$), or may never cross it ($y = x^2 + 1$). It all depends on how many **roots** the quadratic function has.

However, the graph will always have a **y -intercept** — the graph will always cross the y -axis at some point.

✓ Guided Practice

Describe the graphs of the quadratics below in relation to the graph of $y = x^2$.

6. $y = x^2 + 1$

7. $y = x^2 - 3$

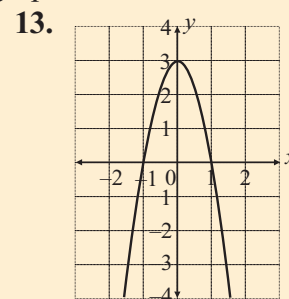
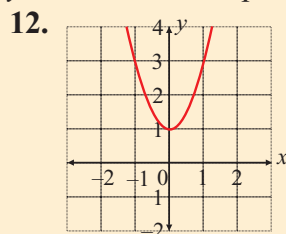
8. $y = 2x^2 + 2$

9. $y = \frac{1}{4}x^2 - 5$

10. $y = -x^2 + 1$

11. $y = -2x^2 - 4$

The graphs in Exercises 12 and 13 are transformations of the graph of $y = x^2$. Find the equation of each graph.



✓ Independent Practice

Match the equations with their graphs on the right.

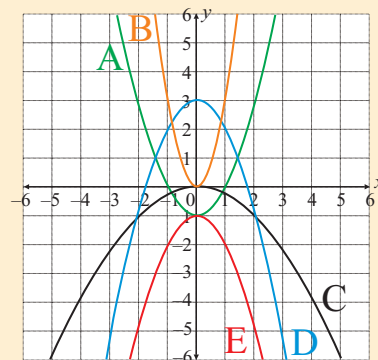
1. $y = x^2 - 1$

2. $y = -x^2 - 1$

3. $y = 3x^2$

4. $y = -\frac{1}{4}x^2$

5. $y = -x^2 + 3$



Describe the graphs of the quadratics below in relation to the graph of $y = x^2$.

6. $y = \frac{1}{2}x^2 + 1$

7. $y = -4x^2$

8. $y = -2x^2 + 3$

9. $y = \frac{1}{3}x^2$

Round Up

Now you know how the a and c parts of the equation $y = ax^2 + c$ affect the graph. In the next Topic you'll learn how to draw some quadratic graphs yourself.

Drawing Graphs of Quadratic Functions

California Standards:

21.0: Students graph quadratic functions and know that their roots are the x -intercepts.

22.0: Students use the quadratic formula or factoring techniques or both to determine whether the graph of a quadratic function will intersect the x -axis in zero, one, or two points.

What it means for you:

You'll graph quadratic functions by finding their roots.

Key words:

- quadratic
- parabola
- intercept
- vertex
- line of symmetry
- root

In this Topic you'll use methods for finding the intercepts and the vertex of a graph to draw graphs of quadratic functions.

Find the Roots of the Corresponding Equations

In general, a good way to graph the function $y = ax^2 + bx + c$ is to find:

- the **x -intercepts** (if there are any)
— this involves solving a quadratic equation,
- the **y -intercept** — this involves setting $x = 0$,
- the **vertex**.

Example 1

Sketch the graphs of $y = x^2 - 3x + 2$ and $y = -2x^2 + 6x - 4$.

Solution

- To find the **x -intercepts** of the graph of $y = x^2 - 3x + 2$, you need to solve: $x^2 - 3x + 2 = 0$

This quadratic factors to give: $(x - 1)(x - 2) = 0$

Using the zero property, $x = 1$ or $x = 2$.

So the x -intercepts are **(1, 0) and (2, 0)**.

- To find the **y -intercept**, put $x = 0$ into $y = x^2 - 3x + 2$.
This gives $y = 2$, so the y -intercept is at **(0, 2)**.

- The x -coordinate of the **vertex** is always **halfway between the x -intercepts**.

So the x -coordinate of the vertex is given by: $x = \frac{1+2}{2} = \frac{3}{2}$

And the y -coordinate of the vertex is: $\left(\frac{3}{2}\right)^2 - \left(3 \times \frac{3}{2}\right) + 2 = -\frac{1}{4}$

So the **vertex is at $\left(\frac{3}{2}, -\frac{1}{4}\right)$** .

Also, the parabola's line of symmetry passes through the vertex.

So here, the line of symmetry is the line **$x = \frac{3}{2}$** .

Example 1 continued

The next function is the same as in the previous example, only multiplied by -2 . The coefficient of x^2 is **negative** this time, so the graph is concave down.

- (i) To find the **x-intercepts** of the graph of $y = -2x^2 + 6x - 4$, you need to solve: $-2x^2 + 6x - 4 = 0$

This quadratic factors to give $-2(x - 1)(x - 2) = 0$.

So using the zero property, $x = 1$ or $x = 2$.

This means the x-intercepts are at **(1, 0) and (2, 0)**.

- (ii) Put $x = 0$ into $y = -2x^2 + 6x - 4$ to find the **y-intercept**.

The y-intercept is **(0, -4)**.

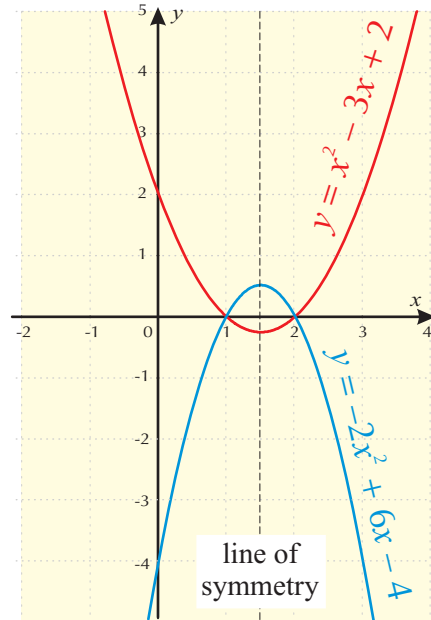
- (iii) The **vertex** is at $x = \frac{3}{2}$.

So the y-coordinate of the **vertex** is at: $-2 \times \left(\frac{3}{2}\right)^2 + \left(6 \times \frac{3}{2}\right) - 4 = \frac{1}{2}$

Therefore the coordinates of the vertex are **$\left(\frac{3}{2}, \frac{1}{2}\right)$** ,

and the line of symmetry is again **$x = \frac{3}{2}$** .

Here are both graphs drawn on the same axes:



Check it out:

In the equation $y = x^2 - 3x + 2$ the coefficient of x^2 is positive ($= 1$), so the parabola will be u-shaped.

✓ Guided Practice

Exercises 1–4 are about the quadratic $y = x^2 - 1$.

1. Find the x -intercepts (if there are any).
2. Find the y -intercepts (if there are any).
3. Find the vertex.
4. Using the vertex, x -intercepts, and y -intercepts, graph the quadratic.

Exercises 5–8 are about the quadratic $y = (x - 1)^2 - 4$.

5. Find the x -intercepts (if there are any).
6. Find the y -intercepts (if there are any).
7. Find the vertex.
8. Using the vertex, x -intercepts, and y -intercepts, graph the quadratic.

✓ Independent Practice

For each of the quadratics in Exercises 1–7, follow these steps:

- i) Find the x -intercepts (if any),
- ii) Find the y -intercepts (if any),
- iii) Find the vertex,
- iv) Using the vertex, x -intercepts, and y -intercepts, graph the quadratic.

1. $y = x^2 - 2x$

2. $y = x^2 + 2x - 3$

3. $y = -4x^2 - 4x + 3$

4. $y = x^2 - 4$

5. $y = x^2 + 4x + 4$

6. $y = -x^2 + 4x + 5$

7. $y = -9x^2 - 6x + 3$

Describe the characteristics of quadratic graphs of the form $y = ax^2 + bx + c$ that have the following features, or say if they are not possible.

8. No x -intercepts

9. One x -intercept

10. Two x -intercepts

11. Three x -intercepts

12. No y -intercepts

13. One y -intercept

14. Two y -intercepts

15. Three y -intercepts

16. Which quadratic equation has the following features?

Vertex $(3, -4)$, x -intercepts $(1, 0)$, $(5, 0)$, and y -intercept $(0, 5)$

17. Which quadratic equation has the following features?

Vertex $(0, 16)$, x -intercepts $(4, 0)$, $(-4, 0)$, and y -intercept $(0, 16)$

Round Up

A quadratic function has the general form $y = ax^2 + bx + c$ (where $a \neq 0$). When you draw the graph of a quadratic, the value of a determines whether the parabola is **concave up** (u-shaped) or **concave down** (n-shaped), and how steep it is. Changing the value of c moves the graph in the direction of the y -axis. Note that if $a = 0$, the function becomes $y = bx + c$, which is a linear function whose graph is a **straight line**.

Quadratic Graphs and Completing the Square

California Standards:

21.0: Students graph quadratic functions and know that their roots are the x -intercepts.

22.0: Students use the quadratic formula or factoring techniques or both to determine whether the graph of a quadratic function will intersect the x -axis in zero, one, or two points.

What it means for you:

You'll graph quadratic functions by first completing the square of the equation.

Key words:

- quadratic
- completing the square
- parabola
- intercept
- vertex
- line of symmetry
- root

If there are *no x -intercepts*, then it's impossible to find the vertex by saying that the vertex is halfway between the x -intercepts (like you saw in Topic 7.4.2).

But you can use the method of *completing the square*.

Write the Equation in the Form $y = (x + k)^2 + p$

Example 1

Sketch the graph of $y = x^2 - 6x + 10$.

Solution

You could try to find the x -intercepts by factoring the equation:

$$x^2 - 6x + 10 = 0$$

This time, the left-hand side doesn't factor.

So to find the solutions you could try the **quadratic formula** with $a = 1$, $b = -6$, and $c = 10$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} \\ &= \frac{6 \pm \sqrt{-4}}{2} \end{aligned}$$

However, since you cannot take the square root of a negative number, this tells you that the quadratic function **has no real roots** — the equation **can't be solved** using real numbers.

This means that the graph of $y = x^2 - 6x + 10$ **never crosses the x -axis**.

But this doesn't mean that you can't find the vertex — you just have to use a different method.

The trick is to write the equation of the quadratic in the form $y = (x + k)^2 + p$ — you need to **complete the square**.

So take the first two terms of the quadratic, and add a number to make a perfect square.

Example 1 continued

$$x^2 - 6x + \left(\frac{-6}{2}\right)^2 = (x - 3)^2$$

$$x^2 - 6x + 10 = (x - 3)^2 - \left(\frac{-6}{2}\right)^2 + 10$$

$$x^2 - 6x + 10 = (x - 3)^2 + 1$$

Therefore the function you need to sketch is $y = (x - 3)^2 + 1$.

Now, the minimum value that $(x - 3)^2$ takes is 0 (since a squared number cannot be negative).

Therefore the **minimum value** of $y = (x - 3)^2 + 1$ is $0 + 1 = 1$.

This minimum value occurs at $x = 3$ (the value for x where $(x - 3)^2 = 0$).

So the coordinates of the **vertex** of the parabola are **(3, 1)**.

As before, the **line of symmetry** passes through the vertex — so the line of symmetry is **$x = 3$** .

The graph of the quadratic function $y = (x + k)^2 + p$ has its vertex at $(-k, p)$.

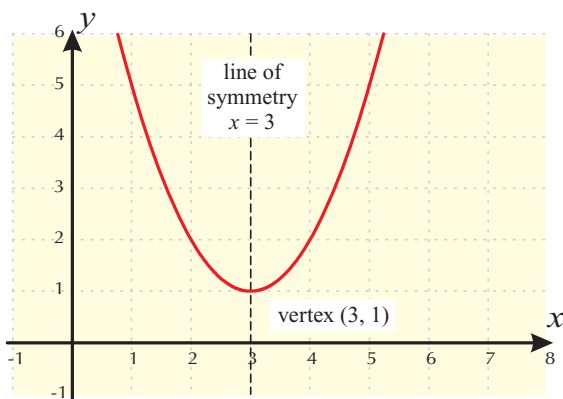
The line of symmetry of the graph is $x = -k$.

Check it out:

The coefficient of x^2 is positive, so this parabola is concave up.

Example 1 continued

The graph of $y = x^2 - 6x + 10 = (x - 3)^2 + 1$:



Find the Vertex by Completing the Square

Example 2

Complete the square for $4x^2 - 12x + 11$. Then find the vertex and line of symmetry of $y = 4x^2 - 12x + 11$.

Solution

This is a **concave-up** parabola, since the coefficient of x^2 is positive.

$$\begin{aligned} 4x^2 - 12x + 11 &= 4\left[x^2 - 3x + \frac{11}{4}\right] \\ &= 4\left[\left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{11}{4}\right] \\ &= 4\left[\left(x - \frac{3}{2}\right)^2 + \frac{1}{2}\right] = 4\left(x - \frac{3}{2}\right)^2 + 2 \end{aligned}$$

Since the minimum value of $\left(x - \frac{3}{2}\right)^2$ is 0, the minimum value of $4\left(x - \frac{3}{2}\right)^2 + 2$ must be $0 + 2 = 2$. This minimum value occurs at $x = \frac{3}{2}$.

So the **vertex** of the graph of $y = 4x^2 - 12x + 11$ is at $\left(\frac{3}{2}, 2\right)$, and the **line of symmetry** is $x = \frac{3}{2}$.

As before, put $x = 0$ to find the **y-intercept** — this is at $y = 11$.

Don't forget:

See Section 7.2 for more about completing the square.

Example 3

Write $4x - x^2 - 7$ in the form $a(x + k)^2 + m$, and sketch the graph.

Solution

$$\begin{aligned} 4x - x^2 - 7 &= -x^2 + 4x - 7 \quad \leftarrow \text{Factor out } -1 \text{ to make} \\ &= -[x^2 - 4x + 7] \quad \text{completing the square easier.} \\ &= -[(x - 2)^2 + 3] \\ &= -(x - 2)^2 - 3 \end{aligned}$$

But $(x - 2)^2$ is never negative — the minimum value it takes is 0.

So $-(x - 2)^2$ can never be **positive**, and the **maximum** value it can take is 0.

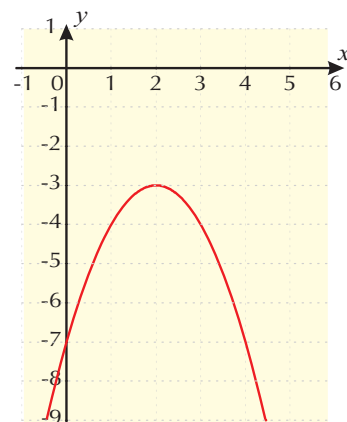
This means that the maximum value of $-(x - 2)^2 - 3$ must be -3 , which it takes when $x - 2 = 0$ — that is, at $x = 2$.

So the **vertex** of the graph is at $(2, -3)$. And the **line of symmetry** is $x = 2$.

As always, find the **y-intercept** by putting $x = 0$. This is at $y = -7$.

Check it out:

The coefficient of x^2 is negative, so this is a concave-down parabola.



✓ Guided Practice

Sketch the graph of each function below, stating the y -intercept and x -intercepts (where appropriate). Use the method of completing the square to find the coordinates of the vertex, and the line of symmetry.

1. $y = x^2 - 12x + 20$

2. $y = x^2 + 8x + 12$

3. $y = x^2 - 2x - 3$

4. $y = x^2 - 4x - 5$

5. $y = -x^2 - 2x + 3$

6. $y = -x^2 - x + 6$

7. $y = -2x^2 - 8x + 10$

8. $y = 2x^2 + x - 6$

9. $y = x^2 - 4x + 12$

10. $y = 3x^2 + 6x + 6$

✓ Independent Practice

In Exercises 1–2, use the information that a ball was thrown vertically into the air from a platform $\frac{3}{2}$ m above sea level. The relationship between the height in meters above sea level, h , and the number of seconds since the ball was thrown, t , was found to be $h = -5t^2 + 6t + \frac{3}{2}$.

1. After how many seconds did the ball reach its maximum height?
2. What was the ball's maximum height above sea level?

The first 8 seconds in the flight of a paper airplane can be modeled by the quadratic $h = \frac{1}{8}t^2 - t + 4$, where h is the height in feet and t is the time in seconds. Use this information to answer Exercises 3–4.

3. In the first 8 seconds of its flight, when did the airplane reach its minimum height?
4. What was the minimum height of the plane in the first 8 seconds of its flight?

A ball is thrown vertically into the air from a platform. The relationship between the ball's height in meters, h , and the number of seconds, t , since the ball was thrown was found to be $h = -5t^2 + 10t + 15$. Use this information to answer Exercises 5–8.

5. After how many seconds did the ball reach its maximum height?
6. What was the maximum height of the ball?
7. At what height was the ball initially thrown?
8. When did the ball hit the ground?

Round Up

Take a look at Section 7.2 if all this stuff about completing the square seems unfamiliar. Completing the square is a really useful way of *graphing quadratics* because it gives you the *vertex* of the graph straightaway.

Section 7.5

Quadratic Equations and the Discriminant

California Standards:

22.0: Students use the quadratic formula or factoring techniques or both to determine whether the graph of a quadratic function will intersect the x -axis in zero, one, or two points.

What it means for you:

You'll learn what the discriminant is, and how you can use it to tell how many roots a quadratic equation has.

Key words:

- discriminant
- quadratic
- intercept
- root

Don't forget:

See Section 7.3 for more on the quadratic formula.

In this Section you'll meet a particular *part* of the *quadratic formula* that tells you *how many solutions* a quadratic equation has.

This Section carries straight on from the previous Section on graphing quadratic functions.

Use the Quadratic Formula to Find x -Intercepts

A general quadratic equation has the form: $ax^2 + bx + c = 0$

The solutions to this equation are given by the **quadratic formula**:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula can be used to help draw the graph of a quadratic function $y = ax^2 + bx + c$.

By finding where $y = 0$ (that is, by solving $ax^2 + bx + c = 0$), you can find the **x -intercepts** of the parabola.

But it's sometimes impossible to get an answer from the quadratic formula. When $b^2 - 4ac$ is **negative**, the **square root** in the formula cannot be taken. This means the graph never crosses the x -axis.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \leftarrow \text{When } b^2 - 4ac < 0, \text{ there are no } x\text{-intercepts — the parabola never crosses the } x\text{-axis.}$$

In fact, the value of $b^2 - 4ac$ tells you a lot about the solutions of a quadratic equation, and about the x -intercepts of the corresponding quadratic graph.

$b^2 - 4ac$ Tells You How Many Roots a Quadratic Has

For the quadratic equation $ax^2 + bx + c = 0$,

the expression $b^2 - 4ac$ is called the **discriminant**.

The discriminant's used to determine the **number of roots** of the function $y = ax^2 + bx + c$ (and so the number of **x -intercepts** of the graph of $y = ax^2 + bx + c$).

$b^2 - 4ac > 0$ Means the Quadratic Has 2 Distinct Roots

If $b^2 - 4ac > 0$ (if $b^2 - 4ac$ is **positive**), then the equation $ax^2 + bx + c = 0$ has **two distinct real solutions**, and the function $y = ax^2 + bx + c$ has **two distinct real roots**.
In other words, the roots are **real** and **unequal**.

Example 1

Describe the nature of the solutions of the equation $2x^2 + 3x - 2 = 0$. Then find the values of the roots, and sketch the graph of $y = 2x^2 + 3x - 2$.

Solution

First write down the values of a , b , and c :

$$2x^2 + 3x - 2 = 0 \quad \text{so } a = 2, b = 3, \text{ and } c = -2$$

So the **discriminant** is: $b^2 - 4ac = 3^2 - [4 \times 2 \times (-2)]$
 $= 9 - (-16) = 9 + 16 = 25$

Since $b^2 - 4ac$ is **positive**, the equation $2x^2 + 3x - 2 = 0$ has **two distinct (unequal) real** solutions.

This in turn means that the function $y = 2x^2 + 3x - 2$ has two real roots — its graph crosses the x -axis in **two** places.

To work out the actual values of the roots, use the **quadratic formula**:

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-3 \pm \sqrt{25}}{4} \\&= \frac{-3 \pm 5}{4} \\&= -2 \text{ or } \frac{1}{2}\end{aligned}$$

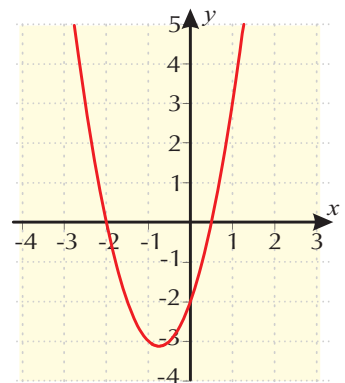
Check it out:

There are two distinct (unequal), real roots — so there are two distinct x -intercepts on the graph.

So the graph of $y = 2x^2 + 3x - 2$ meets the x -axis in two places: $(-2, 0)$ and $(\frac{1}{2}, 0)$.

To sketch the graph, you also need the **y -intercept**:

$$y = 2x^2 + 3x - 2 = 2(0)^2 + 3(0) - 2 = -2$$



Example 2

Find the nature of the solutions of the quadratic equation $x^2 + 5x + 2 = 0$.

Solution

Again, the best thing to do first is write down the values of a , b , and c :

$$x^2 + 5x + 2 = 0 \quad \text{so } a = 1, b = 5, \text{ and } c = 2$$

Then the discriminant is:

$$\begin{aligned} b^2 - 4ac &= 5^2 - (4 \times 1 \times 2) \\ &= 25 - 8 \\ &= 17 \end{aligned}$$

The discriminant is **positive**, so there are **two unequal, real solutions** of $x^2 + 5x + 2 = 0$.

You could use the quadratic formula to find the actual solutions:

$$x = \frac{-5 \pm \sqrt{17}}{2} = \frac{-5 + \sqrt{17}}{2} \text{ or } \frac{-5 - \sqrt{17}}{2}$$

So the x -intercepts of the graph of $y = x^2 + 5x + 2$ are at:

$$\left(\frac{-5 + \sqrt{17}}{2}, 0 \right) \text{ and } \left(\frac{-5 - \sqrt{17}}{2}, 0 \right)$$

 **Guided Practice**

Describe the nature of the solutions of each quadratic equation, and find the values of the solutions.

1. $x^2 + x - 12 = 0$

2. $x^2 + 2x - 3 = 0$

3. $x^2 + 5x + 4 = 0$

4. $3x^2 - 7x + 4 = 0$

5. $2x^2 + 5x + 2 = 0$

6. $x^2 + 3x - 1 = 0$

7. $3x^2 + 7x - 6 = 0$

8. $2x^2 + 9x = 5$

9. $2(5x^2 + 1) = 9x$

10. $6(2x^2 + 1) + 17x = 0$

11. $x(10x + 7) = -1$

12. $2x(4x + 7) = -3$

Use the x - and y -intercepts of the quadratics below to decide in which quadrant the vertex of each equation would be.

13. $y = x^2 + x - 12$

14. $y = x^2 + 2x - 3$

15. $y = -x^2 + 2x + 3$

16. $y = 3x^2 - 12x - 15$

$b^2 - 4ac = 0$ Means the Quadratic Has 1 Double Root

If $b^2 - 4ac = 0$, then the solutions of $ax^2 + bx + c = 0$ (and the roots of $y = ax^2 + bx + c$) are **real** and **equal**.

Graphically, **equal roots** (also called a **double root**) mean the graph of the quadratic function just **touches** the x -axis, but does not cross it.

Example 3

Determine the nature, and find the value(s), of the solution(s) of the equation $x^2 - 6x + 9 = 0$. Sketch the graph of $y = x^2 - 6x + 9$.

Solution

First write down the values of a , b , and c :

$$x^2 - 6x + 9 = 0 \quad \text{so } a = 1, b = -6, \text{ and } c = 9$$

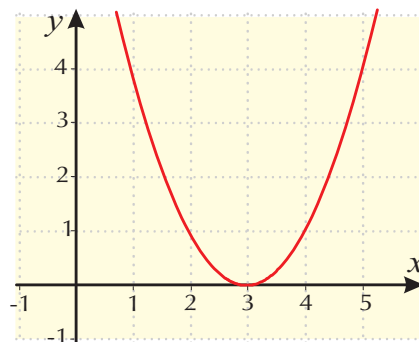
So the **discriminant** is:

$$\begin{aligned} b^2 - 4ac &= (-6)^2 - 4 \times 1 \times 9 \\ &= 36 - 36 \\ &= 0 \end{aligned}$$

This time, the discriminant is **zero**, so $y = x^2 - 6x + 9$ has a **double root**. In other words, its graph just touches the x -axis without actually crossing it.

As always, to work out where this double root is, use the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-6)}{2 \cdot 1} \\ &= \frac{6}{2} \\ &= 3 \end{aligned}$$



So the graph of $y = x^2 - 6x + 9$ touches the x -axis at **(3, 0)**.

Check it out:

$b^2 - 4ac = 0$, so the quadratic formula simplifies to $\frac{-b}{2a}$.

✓ Guided Practice

Describe the nature of the solutions of the quadratic equations, and find the value(s) of the solution(s).

17. $x^2 + 4x + 4 = 0$

19. $x^2 + 6x = -9$

21. $5x(5x - 6) = -9$

23. $4x^2 = -5(4x + 5)$

25. $8x(x + 3) + 18 = 0$

18. $9x^2 - 6x + 1 = 0$

20. $3x(x - 10) + 75 = 0$

22. $8x(2x + 3) = -9$

24. $4x(x + 1) + 1 = 0$

26. $24x(2x + 9) + 243 = 0$

$b^2 - 4ac < 0$ Means the Quadratic Has No Real Roots

If $b^2 - 4ac < 0$ (if $b^2 - 4ac$ is **negative**), then $ax^2 + bx + c = 0$ has **no real solutions**, and $y = ax^2 + bx + c$ has **no real roots**.

If there are **no real roots**, then there are **no x-intercepts** — the graph of the quadratic does not intersect (cross or touch) the x -axis.

Example 4

Describe the graphs of: (a) $y = x^2 + 2x + 3$, and (b) $y = -2x^2 + 4x - 5$

Solution

(a) Here $a = 1$, $b = 2$, and $c = 3$, so:

$$\begin{aligned}b^2 - 4ac &= 2^2 - 4 \times 1 \times 3 \\ &= 4 - 12 = \mathbf{-8}\end{aligned}$$

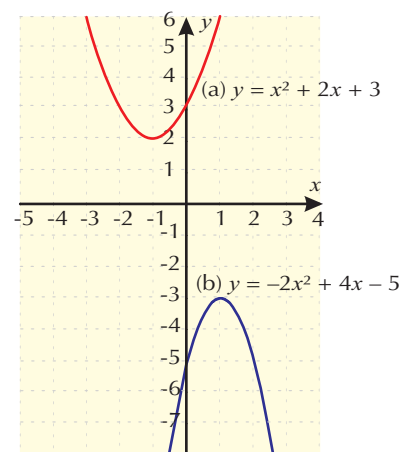
So the graph of $y = x^2 + 2x + 3$ **never intersects the x -axis**. Since $a > 0$, the graph is **concave up** (u-shaped) and stays **above** the x -axis.

(b) Here $a = -2$, $b = 4$, and $c = -5$, so:

$$\begin{aligned}b^2 - 4ac &= 4^2 - 4 \times (-2) \times (-5) \\ &= 16 - 40 = \mathbf{-24}\end{aligned}$$

So the graph of $y = -2x^2 + 4x - 5$ **never intersects the x -axis** either.

But this time, since $a < 0$, the graph is **concave down** (n-shaped) and stays **below** the x -axis.



✓ Guided Practice

Use the discriminant to verify that there are no real number solutions for the quadratic equations below.

27. $x^2 + 3x + 4 = 0$

28. $5x^2 - 4x + 3 = 0$

29. $3x^2 + 8 = 9x$

30. $2x^2 = 7(x - 1)$

31. $2x(2x + 3) + 3 = 0$

32. $3x^2 + 1 = 3x$

33. $3x^2 + 5x + 3 = 0$

34. $2x(x - 3) = -5$

Using the Discriminant

Example 5

Suppose the graph of $y = kx^2 + x - 6$ intersects the x -axis at two distinct points (where k is some constant). What are the possible values of k ?

Solution

Since the graph of $y = kx^2 + x - 6$ has two distinct x -intercepts, **the discriminant** ($b^2 - 4ac$) **must be positive**.

So write down your a , b , and c : $a = k$, $b = 1$, $c = -6$

Now use the fact that the discriminant is positive:

$$\begin{aligned}b^2 - 4ac &> 0 \\1^2 - 4k(-6) &> 0 \\1 + 24k &> 0 \\24k &> -1 \\k &> -\frac{1}{24}\end{aligned}$$

So k can be any real number greater than $-\frac{1}{24}$, but not zero:

$$-\frac{1}{24} < k < 0 \text{ or } k > 0.$$

Check it out:

If $k = 0$, the x^2 term would disappear, leaving a linear equation. The graph crosses the x -axis in two places, so the equation can't be linear.

Example 6

At how many points does the graph of $y = x^2 - 2x + 3$ intersect the x -axis?

Solution

Here, $a = 1$, $b = -2$, $c = 3$

$$\begin{aligned}\text{So } b^2 - 4ac &= (-2)^2 - 4 \times 1 \times 3 \\&= 4 - 12 = -8\end{aligned}$$

The discriminant is **negative**, so $y = x^2 - 2x + 3 = 0$ has **no real roots**, and so the graph of $y = x^2 - 2x + 3$ **does not intersect the x -axis**.

Example 7

Find the values of k for which $y = 5x^2 - 3x + k$ has a double root.

Solution

As always, it's a good idea to begin by writing down your a , b , and c :

$$a = 5, b = -3, c = k$$

$$\begin{aligned}\text{So } b^2 - 4ac &= (-3)^2 - 4 \times 5 \times k \\&= 9 - 20k\end{aligned}$$

The quadratic has a **double root** where the **discriminant** equals **zero**.

This is the case when $9 - 20k = 0$, or when $k = \frac{9}{20}$.

So $y = 5x^2 - 3x + k$ has a double root only when $k = \frac{9}{20}$.

Example 8

Describe the nature of the roots of $y = x^2 + \sqrt{2}x + \frac{1}{2}$.

Solution

This time, $a = 1$, $b = \sqrt{2}$, and $c = \frac{1}{2}$,

$$\text{so } b^2 - 4ac = (\sqrt{2})^2 - 4 \times 1 \times \frac{1}{2} = 2 - 2 = 0$$

The discriminant is **zero**, so the function has a **double root**.

$$\text{This root is at } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} = \frac{-\sqrt{2}}{2 \cdot 1} = \frac{-\sqrt{2}}{2}$$

Independent Practice

Determine the number of roots of the following functions, and find the values of any real roots.

1. $y = x^2 - 2x - 3$

2. $y = x^2 + 4x + 3$

3. $y = x^2 - 3x - 1$

4. $y = x^2 + x - 1$

5. $y = 4x^2 - 4x + 1$

6. $y = x^2 - 8x + 16$

7. $y = 2x^2 - 4x + 3$

8. $y = x^2 + x + 3$

9. $y = 2x^2 + x - 6$

10. $y = 4x^2 - 12x + 9$

11. $y = x^2 - x + 5$

12. $y = x^2 - 2x - 2$

13. Find the possible values of k if $y = 3x^2 - kx + 3$ has a double root.

14. Find the possible values of p if $2x^2 - 5x + p = 0$ has two real solutions.

15. If $x = -1$ is a root of $y = 3x^2 + x - k$, find k and the other root.

16. If $x = \frac{1}{2}$ is a solution to $kx^2 + 9x - 5 = 0$, find both solutions.

17. Find the possible values of p if $y = px^2 - 7x - 7$ has no real roots.

State the number of times that the graphs of the following quadratic functions intercept the x -axis:

18. $y = x^2 - 3x - 28$

19. $y = 4x^2 + 4x + 1$

20. $y = 4x^2 + 2x + 1$

21. $y = 2x^2 - x - 1$

22. $y = 5x^2 + 3x + 1$

23. Find all possible values of k when $y = 3x^2 - 2kx + k$ has a double root.

24. When $y = 3x^2 - 2kx + k$ has a root of $x = 2$, find k and the other root.

Round Up

Remember — the discriminant is just one part of the quadratic formula. If all you need to know is how many roots a function has, you don't need to use the full formula — just the discriminant.

Topic 7.6.1

Section 7.6 Motion Tasks

California Standards:

23.0: Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.

What it means for you:

You'll model objects under the force of gravity using quadratic equations, and then solve the equations.

Key words:

- quadratic
- gravity
- vertex
- parabola
- intercept
- completing the square

Quadratic equations have *applications in real life*. In particular, you can use them to model objects that are *dropped or thrown up into the air*.

Quadratic Functions Describe Motion Under Gravity

Example 1

The height of a stone thrown up in the air is modeled by the equation $h = 80t - 16t^2$, where t represents the time in seconds since the stone was thrown and h is the height of the stone in feet.

After how many seconds is the stone at a height of 96 feet? Explain your answer.

Solution

The stone reaching a height of 96 feet is represented by $h = 96$, so you need to solve $80t - 16t^2 = 96$.

Rewriting this in the form $ax^2 + bx + c = 0$ (using t instead of x) gives:

$$16t^2 - 80t + 96 = 0$$

$$t^2 - 5t + 6 = 0$$

Divide through by 16

$$(t - 2)(t - 3) = 0$$

Factor the quadratic equation

$$t - 2 = 0 \text{ or } t - 3 = 0$$

Solve using the zero property

$$t = 2 \text{ or } t = 3$$

So the stone is at a height of 96 feet after **2 seconds** (on the way **up**), and again after **3 seconds** (on the way **down**).

Example 2

Use the same information from Example 1. After how many seconds does the stone hit the ground? Explain your answer.

Solution

When the stone hits the ground, $h = 0$. So solve $80t - 16t^2 = 0$.

$$5t - t^2 = 0$$

Divide through by 16

$$t(5 - t) = 0$$

$$t = 0 \text{ or } t = 5$$

Solve using the zero property

But $t = 0$ represents when the stone was **thrown**, so the stone must land at $t = 5$ — **after 5 seconds**.

Example 3

Use the same information from Example 1.
Calculate h at $t = 7$. Explain your answer.

Solution

$$\text{At } t = 7, h = (80 \times 7) - (16 \times 7^2) = 560 - 784 = -224$$

Negative values of h suggest that the stone is **beneath** ground level.
This can't be true — the height can't be less than zero feet.

But the stone landed after 5 seconds — so after $t = 5$, the function $h = 80t - 16t^2$ **doesn't describe the motion of the stone**.

Check it out:

The formula $h = 80t - 16t^2$ only describes the stone's motion for a certain range of t . It can't describe what happens before the stone is thrown or after it lands.

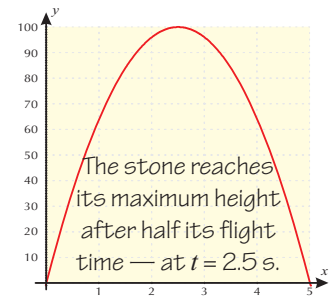
Example 4

Use the same information from Example 1.
What is the maximum height of the stone? Justify your answer.

Solution

The graph of $h = 80t - 16t^2$ is a **parabola**, and the maximum height reached is represented by the **vertex** of the parabola, which you can find by **completing the square**.

$$\begin{aligned} h &= 80t - 16t^2 \\ &= -16(t^2 - 5t) \\ &= -16\left[\left(t - \frac{5}{2}\right)^2 - \frac{25}{4}\right] \\ &= -16\left(t - \frac{5}{2}\right)^2 + 100 \end{aligned}$$



So the maximum height is **100 feet** (which is reached at $t = \frac{5}{2} = 2.5$ s).

✓ Guided Practice

1. In a Physics experiment, a ball is thrown into the air from an initial height of 24 meters. Its height h (in meters) at any time t (in seconds) is given by $h = -5t^2 + 10t + 24$.

Find the maximum height of the ball and the time t at which it will hit the ground.

2. A firework is propelled into the air from the ground. Its height after t seconds is modeled by $h = 96t - 16t^2$. The firework needs to explode at a height of 128 feet from the ground.

After how long will it first reach this height? If the firework fails to explode, when will it hit the ground?

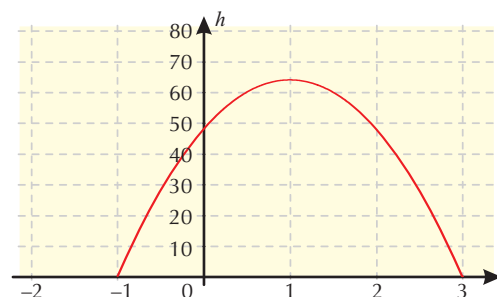
Example 5

The height above the ground in feet (h) of a ball after t seconds is given by the quadratic function $h = -16t^2 + 32t + 48$.

Explain what the h -intercept and t -intercepts mean in this situation, and find the maximum height reached by the ball.

Solution

To get a clearer picture of what everything means, it helps to draw a **graph**.



The intercept on the **vertical axis** (the h -axis) is found by putting $t = 0$: $h = 48$.

The intercepts on the **horizontal axis** (the t -axis) are found by solving $h = 0$ — that is, $-16t^2 + 32t + 48 = 0$.

$$\begin{aligned} t^2 - 2t - 3 &= 0 \\ (t - 3)(t + 1) &= 0 \\ t = 3 \text{ or } t = -1 \end{aligned}$$

In this situation, the intercept on the vertical axis (the h -intercept) represents the **initial height** of the ball when it was thrown (at $t = 0$). So here, the **ball was thrown from 48 feet above the ground**.

The intercepts on the horizontal axis (the t -intercepts) represent the times at which the ball was **at ground level**. However, the function only describes the motion of the ball between $t = 0$ (when it was thrown) and $t = 3$ (when it lands).

So the **t -intercept at $t = 3$ represents the point when the ball lands**. The **t -intercept at $t = -1$ doesn't have any real-life significance here**.

To find the **maximum height**, you need to find the **vertex** of the parabola — so **complete the square**:

$$\begin{aligned} -16t^2 + 32t + 48 &= -16[t^2 - 2t - 3] \\ &= -16[(t - 1)^2 - 4] \\ &= -16(t - 1)^2 + 64 \end{aligned}$$

The vertex of this parabola occurs where $t = 1$, and so the vertex is at $(1, 64)$. This means the **maximum height of the ball is 64 feet**.

Check it out:

From the graph, you can see that the ball reaches its maximum height before half its flight time has passed.

Check it out:

Put $t = 1$ into the quadratic to find the maximum value.

Independent Practice

A baseball is hit from homebase. Its height in meters is modeled by the equation $h = 25t - 5t^2$, where t is the time in seconds.

1. After how many seconds will the ball be at a height of 20 meters?
2. What height will the baseball reach before it starts descending?

A rocket is fired into the air. Its height in feet at any time is given by the equation $h = 1600t - 16t^2$, where t is the time in seconds.

3. Find the height of the rocket after 2 seconds.
4. After how many seconds will the rocket be 30,000 feet above the ground?
5. After how many seconds will the rocket hit the ground?

6. As a skydiver steps out of a plane, she drops her watch. The distance in feet, h , that the watch has fallen after t seconds is given by the equation $h = 16t^2 + 4t$. After how many seconds will the watch have fallen 600 feet?

The height in feet of an object projected upwards is modeled by the equation $h = 100t - 16t^2$.

7. How long after being projected is the object 100 feet above the ground?
8. What is the greatest height reached by the object?

The area of a rectangle is given by the formula $A = x(20 - x)$ cm, where x is the width.

9. Find x when $A = 84$ cm².
10. What value of x maximizes the area A ?

James and Mei are each standing on diving boards, and each throw a ball directly upwards. The height of each ball above the pool in feet, h , is plotted against the time in seconds, t , since it was thrown.

11. The height of James's ball can be calculated using the equation $h = -16t^2 + 30t + 10$. From what height above the pool does James throw his ball?
12. The height of Mei's ball can be calculated using the equation $h = -16t^2 + 32t + 20$. After how many seconds does her ball reach its maximum height?
13. Calculate the difference in maximum heights of the balls, to 1 decimal place.

Round Up

Plotting a graph is often a good idea when you're working on motion problems — because then you can see at a glance when an object reaches the ground, or when it reaches its maximum height.

Economic Tasks

California Standards:

23.0: Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.

What it means for you:

You'll model money problems using quadratic equations, and then solve the equations.

Key words:

- economic
- profit
- quadratic
- vertex
- parabola
- completing the square

As well as the motion tasks you saw in Topic 7.6.1, you can use quadratic equations to *model real-life problems involving money*.

Applications of Quadratics to Economics

The best way to introduce quadratic equations modeling money problems is to show you an example:

Example 1

The owner of a restaurant wishes to graph the annual profit of his restaurant against the number of people he employs. He calculates that the annual profit in thousands of dollars (P) can be modeled by the formula $P = -0.3x^2 + 4.5x$, where x is the number of people employed.

According to the owner's formula, how many full-time members of staff does the restaurant have to employ to make a profit of \$15,000?

Solution

You have a formula for the profit P , and you have to find when this equals 15 (since the formula gives you the profit in **thousands of dollars**). So you need to solve the quadratic equation $-0.3x^2 + 4.5x = 15$.

Rewriting this in the form $ax^2 + bx + c = 0$ gives:

$$0.3x^2 - 4.5x + 15 = 0$$

$$x^2 - 15x + 50 = 0$$

$$(x - 10)(x - 5) = 0$$

$$x = 10 \text{ or } x = 5$$

Divide through by 0.3

Solve using the zero property

This means that the restaurant can employ either **5 people** or **10 people** and make a profit of \$15,000.

Example 2

Use the same information from Example 1.

According to the owner's formula, how many full-time members of staff should the restaurant employ to make maximum profit?

Solution

To find the **maximum profit**, you need to find the **maximum value** of the **quadratic** $P = -0.3x^2 + 4.5x$.

To do this, you can **complete the square**:

$$\begin{aligned} P &= -0.3x^2 + 4.5x \\ &= -0.3(x^2 - 15x) \\ &= -0.3\left[\left(x - \frac{15}{2}\right)^2 - \left(\frac{15}{2}\right)^2\right] \\ &= -0.3\left(x - \frac{15}{2}\right)^2 + 0.3\left(\frac{225}{4}\right) \\ &= -0.3\left(x - \frac{15}{2}\right)^2 + \frac{3}{10}\left(\frac{225}{4}\right) \\ &= -0.3\left(x - \frac{15}{2}\right)^2 + \frac{135}{8} \end{aligned}$$

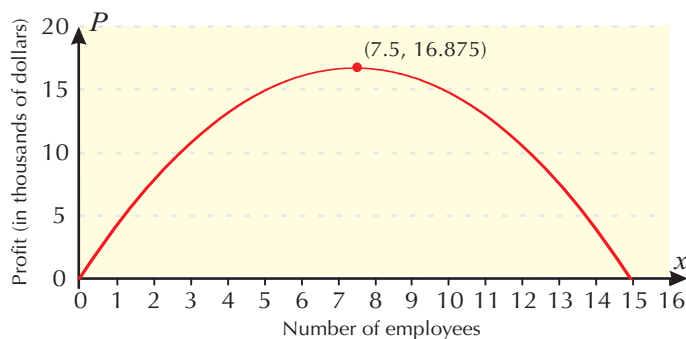
So the vertex of the parabola is at $\left(\frac{15}{2}, \frac{135}{8}\right)$, which (in theory) means that the restaurant should employ **7.5 people** to make the maximum possible profit of \$16,875.

Clearly, the restaurant can't employ 7.5 people — a good idea now is to draw the graph so that you can answer this question more realistically.

Find the x -intercepts by solving $P = 0$:

$$\begin{aligned} P &= -0.3x^2 + 4.5x \\ &= -0.3(x^2 - 15x) \\ &= -0.3x(x - 15) \\ &= 0 \text{ at } x = 0 \text{ and } x = 15 \end{aligned}$$

So the graph looks like this:



Check it out:

Work out the profit like this:

$$\begin{aligned} &(-0.3 \times 7^2) + (4.5 \times 7) \\ &= (-0.3 \times 8^2) + (4.5 \times 8) \\ &= 16.8 \end{aligned}$$

Example 2 continued

You can see from the **symmetry** of the graph (the **line of symmetry** is $x = 7.5$) that the maximum possible profit while employing a whole number of people is at $x = 7$ and $x = 8$, at which points the profit is **\$16,800**.

So, if the restaurant employs more than 8 people, profits decrease, possibly because there is not enough work for more than 8 people to do efficiently.

✓ Guided Practice

1. The profit p in cents per 10-minute period earned from driving a taxicab is given by $p = 80x - 3x^2$, where x is the speed in mph. What speed would yield a profit of 512 cents per 10-minute period?
2. An investor kept track of her portfolio profit, P , at time, t , measured in years after she began investing. If $P = 4000t^2 - 28000t + 3000$ represents her profit, after how many years will she have made \$150,000 profit?
3. The amount of money a customer is willing to spend at a store is related to t , the number of minutes they have to wait before being served. If $M = -t^2 + 8t + 17$ represents the money a customer spends, how long will it take before a customer decides to leave the store without spending any money?

✓ Independent Practice

Leo produces x pounds of salsa. The ingredients cost $0.1x^2 - 30$ dollars and he makes $2x$ dollars revenue from the sale of his salsa.

1. What is Leo's maximum possible profit?
2. How many pounds of salsa would Leo need to sell to break even?

The value in dollars, V , of a certain stock can be modeled by the equation $V = -16t^2 + 88t + 101$, where t represents the time in months.

3. What was the original value of the stock?
4. What was the maximum value of the stock?
5. When did the stock reach the maximum value?
6. When did the stock become worthless?

The value, V , of Juan's investment portfolio can be modeled by the equation $V = 16t^2 - 256t + 16,000$, where t is the time in months.

7. What was the original value of Juan's portfolio?
8. What was the minimum value of Juan's portfolio?
9. When will Juan's investment portfolio be worth \$16,576.00?

Round Up

Usually when you graph quadratic problems involving **money**, the **vertex** of the graph shows you the point where there's **maximum profit**.

Chapter 7 Investigation

The Handshake Problem

This Investigation shows that Math techniques can be useful everywhere — even at parties.

There are **45 people** at a party. If each person shakes hands with every other person, **how many handshakes** will there be altogether?

Suppose there are n people at a party. How many handshakes will there be in terms of n ?

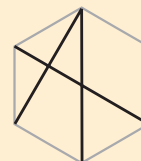


Things to think about:

- How many handshakes does each person make?
- How many handshakes would there be if there were **smaller groups** of people? Are there any **patterns**?

Extension

- 1) The **triangular numbers** form the following sequence: **1, 3, 6, 10, 15, 21, ...**
If the **first** triangular number is 1, what will the **n th** triangular number be?
- 2) There are n people at a round table.
Each person shakes hands with everyone else at the table, **except** for the person on their left and the person on their right. How many handshakes will there be in total?
- 3) Three **diagonals** of a **hexagon** are shown on the right.
(The sides of the shape are **not** diagonals.)
How many diagonals will a regular polygon with n sides have?
Explain your answer.



Open-ended Extension

- 1) At a high school reunion **everyone** shakes hands with **everyone else** once.
Towards the end of the evening, a math teacher arrives and shakes hands with **just** the students that he actually taught. There were **3107 handshakes** in total.

How many people were at the reunion before the math teacher arrived and how many of the people present had the math teacher taught?

Is your answer the **only** one possible? Explain your reasoning.
- 2) Look back at the original **handshake problem**. **Change** the problem in some way and **investigate** the effect on the number of handshakes.

For example, you could investigate the number of handshakes if each person shakes **both** the left and right hands of each other person with their own left and right hand.
How many handshakes would there be for groups of extraterrestrials with h hands?

Round Up

In this Investigation you've recorded results, identified patterns and applied what you've learned to new situations. There's a lot you can say about even quite simple-looking math problems.

Chapter 8

Rational Expressions and Functions

Section 8.1	Rational Expressions	388
Section 8.2	Multiplying and Dividing Rational Expressions	394
Section 8.3	Adding and Subtracting Rational Expressions	403
Section 8.4	Solving Equations with Fractions	412
Section 8.5	Relations and Functions	420
Investigation	Transforming Functions	435

Topic 8.1.1

California Standards:

12.0: Students simplify fractions with polynomials in the numerator and denominator by factoring both and reducing them to the lowest terms.

What it means for you:

You'll find out about the conditions for rational numbers to be defined.

Key words:

- rational
- numerator
- denominator
- undefined

Don't forget:

Take another look at Chapter 1 to remind yourself about rational numbers.

Check it out:

p is called the numerator and q is called the denominator.

Section 8.1

Fractions and Rational Expressions

In this Topic you'll find out about the *necessary conditions* for rational numbers to be defined.

Rational Expressions Can Be Written as Fractions

A rational expression is any **expression** that can be written in the form of a fraction — that means it has a **numerator** and a **denominator**.

Examples of rational expressions are $\frac{3}{1}$, $\frac{3}{4}$, $\frac{8}{9}$, $\frac{1}{x+1}$, $\frac{x-1}{2x+1}$.

Rational expressions are **written in the form** $\frac{p}{q}$, where $q \neq 0$.

An Expression is Undefined if the Denominator is Zero

If the denominator is **equal to zero**, then the expression is said to be **undefined** (see Topic 1.3.4).

So, for example, $\frac{x-1}{2x+1}$ is defined whenever x is not equal to -0.5 .

Example 1

Determine the value of x for which the expression $\frac{7}{x+2}$ is undefined.

Solution

It is undefined when the denominator $x + 2$ equals zero.

This means that the expression is undefined when $x = -2$.

Example 2

Determine the value(s) of x for which the expression $\frac{2}{x^2-4}$ is undefined.

Solution

It's undefined when the denominator $x^2 - 4$ equals zero.

So, solve $x^2 - 4 = 0$ to find the values of x :

$$\begin{aligned}x^2 - 4 &= (x - 2)(x + 2) = 0 \\x - 2 &= 0 \quad \text{or} \quad x + 2 = 0 \\ \Rightarrow x &= 2 \quad \text{or} \quad x = -2\end{aligned}$$

Therefore, $\frac{2}{x^2-4}$ is undefined when $x = \pm 2$.

Example 3

Determine the value(s) of x for which the expression

$$\frac{7x}{x^2 - 8x + 15} \text{ is undefined.}$$

Solution

Factor the denominator to give: $\frac{7x}{(x-3)(x-5)}$

If the denominator equals zero, the expression is undefined.
This happens when either $(x-3)$ or $(x-5)$ equals zero.

So, the expression is undefined when $x = 3$ or $x = 5$.

Guided Practice

Determine the value(s) of the variables that make the following rational expressions undefined.

1. $\frac{3}{4y-1}$

2. $\frac{3-2x}{8+4x}$

3. $\frac{x+1}{5x-30}$

4. $\frac{a^2-2a+1}{a^2+7a+12}$

5. $-\frac{2}{4y^2+11y-3}$

6. $\frac{4k+5}{26+11k-k^2}$

7. $\frac{x^2+1}{x^3-9x}$

8. $\frac{2x+1}{x^2-3x-28}$

9. $\frac{2a^3+a^2}{3a^3-6a^2-45a}$

Independent Practice

Determine the value(s) of the variables that make the following rational expressions undefined.

1. $\frac{x-4-x^2}{3x-6}$

2. $\frac{y^3+y}{3y^2+9y+6}$

3. $\frac{m+5}{m^2+8m+7}$

4. $\frac{5x}{6x-x^2-x^3}$

5. $\frac{2k-1}{3k^3-3k^2-18k}$

6. $\frac{6x^2+13x+5}{8x^3+32x^2+30x}$

7. Jane states that the rational expression $\frac{3x+1}{3x^2-2x-1}$ is defined when x is any real number. Show that Jane is incorrect.

Round Up

This Topic about the *limitations on rational numbers* will help you when you're dealing with *fractions* in later Topics. In Topic 8.1.2 you'll simplify rational expressions to their *lowest terms*.

Topic 8.1.2

Equivalent Fractions

California Standards:

12.0: Students simplify fractions with polynomials in the numerator and denominator by factoring both and reducing them to the lowest terms.

What it means for you:

You'll learn about equivalent fractions and how to simplify fractions to their lowest terms.

Key words:

- equivalent
- rational
- simplify
- common factor

Check it out:

The two shapes on the far right represent x being divided up into the fractions $\frac{5}{6}$ and $\frac{10}{12}$.

Saying that two rational expressions are **equivalent** is just a way of saying that two fractions represent the **same thing**.

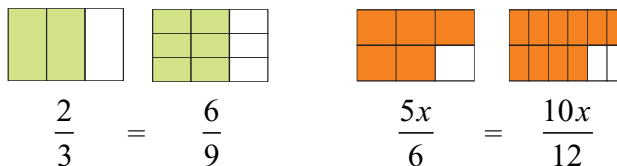
Equivalent Fractions Have the Same Value

A **ratio** is a comparison of two numbers, often expressed by a fraction — for example, $\frac{a}{b}$.

A **proportion** is an **equality** between two **ratios**.

Four quantities a , b , c , and d are in proportion if $\frac{a}{b} = \frac{c}{d}$.

Fractions like these that represent the same rational number or expression are often called **equivalent fractions**.



You can determine whether two fractions are equivalent by using this rule:

The **rational expressions** $\frac{a}{b}$ and $\frac{c}{d}$ are **equivalent** if $ad = bc$.

Example 1

Prove that $\frac{5x}{6}$ and $\frac{10x}{12}$ are equivalent.

Solution

$5x \cdot 12 = 60x$ **This is ad in the rule above**
 and $6 \cdot 10x = 60x$ **This is bc in the rule above**

So, the two rational expressions are equivalent.

✓ Guided Practice

Prove that the following pairs of rational expressions are equivalent.

- $\frac{54m}{6}$ and $\frac{18m}{2}$
- $\frac{1}{3}$ and $\frac{2x}{6x}$
- $\frac{12}{3x-9}$ and $\frac{4}{x-3}$

Simplify Fractions by Canceling Common Factors

A rational expression can be written in its **lowest terms** by reducing it to the **simplest equivalent fraction**. This is done by **factoring** both the numerator and denominator and then **canceling** the common factors — that means making sure its numerator and denominator have **no common factors** other than **1**.

Check it out:

All of the common factors have been canceled, leaving the simplest equivalent fraction.

Don't forget:

The GCF of two numbers is the largest number that is a factor of both of them.

Check it out:

$\frac{7}{8x}$ is the simplest form of $\frac{56}{64x}$ because 1 is the only common factor of the numerator and denominator.

For example:
$$\frac{66}{78} = \frac{\cancel{6} \cdot 11}{\cancel{6} \cdot 13} = \frac{11}{13}$$

Example 2

Reduce the expression $\frac{56}{64x}$ to its lowest terms.

Solution

The greatest common factor (GCF) of 56 and 64 is 8.

This means that:
$$\frac{56}{64x} = \frac{7 \cdot \cancel{8}}{(8 \cdot \cancel{8})x} = \frac{7}{8x}$$

So, $\frac{56}{64x}$ and $\frac{7}{8x}$ are **equivalent fractions**.

Numbers are not the only things that can be canceled — **variables** can be canceled too.

For example:
$$\frac{mc}{cv} = \frac{\cancel{m} \cdot c}{\cancel{c} \cdot v} = \frac{m}{v}$$

Guided Practice

Reduce each of the following rational expressions to their lowest terms.

4. $\frac{21}{28}$

5. $\frac{12}{18}$

6. $\frac{bx}{b^2x}$

7. $\frac{10d}{30}$

8. $-\frac{4m^2c}{10m^3c^2}$

9. $\frac{b(b-3)}{(b-3)}$

10. $\frac{(5+m)(5-m)}{(3+m)(5+m)}$

11. $\frac{x+5}{3x+15}$

12. $\frac{m^2(m+4)(m-4)}{m(m-4)}$

Some Harder Examples to Think About

Factoring the **numerator** and **denominator** is the key to doing this type of question. **Breaking down** a complicated expression into its **factors** means you can **spot the terms** that will cancel.

Check it out:

The greatest common factor of $(x + 3)$ and 6 is 1.

So, $\frac{x+3}{6}$ is the simplest form.

Example 3

Simplify the expression $\frac{x^2 - 9}{6x - 18}$.

Solution

Factor the numerator and denominator, then cancel common factors:

$$\frac{x^2 - 9}{6x - 18} = \frac{\cancel{1}(x-3)(x+3)}{6\cancel{(x-3)}_1} = \frac{x+3}{6}$$

Example 4

Simplify the expression $\frac{9 - m^2}{m^2 - m - 6}$.

Solution

Factor the numerator and denominator completely.

$$\begin{aligned} \frac{9 - m^2}{m^2 - m - 6} &= \frac{(3 - m)(3 + m)}{(m - 3)(m + 2)} \\ &= \frac{\cancel{1}(3 - m)(3 + m)}{-1 \cdot \cancel{(3 - m)}_1(m + 2)} \\ &= \frac{(3 + m)}{-1 \cdot (m + 2)} && \text{Cancel the common factor } (3 - m) \\ &= -\frac{m + 3}{m + 2} \end{aligned}$$

Check it out:

Note that $m - 3 = -1(3 - m)$ so you can substitute this into the expression.

Example 5

Reduce this expression to its lowest terms: $\frac{x^3 - 2x^2 - 15x}{x^3 + 10x^2 + 21x}$

Solution

Factor both the numerator and denominator.

$$\begin{aligned} \frac{x^3 - 2x^2 - 15x}{x^3 + 10x^2 + 21x} &= \frac{x \cdot (x^2 - 2x - 15)}{x \cdot (x^2 + 10x + 21)} \\ &= \frac{\cancel{1}x \cdot \cancel{(x-5)}(x+3)^1}{\cancel{1}x \cdot \cancel{(x+3)}(x+7)} && \text{Cancel the common factors} \\ &= \frac{x - 5}{x + 7} \end{aligned}$$

✓ Guided Practice

13. Show how you can simplify the rational expression $\frac{a^2 + 5a + 6}{a^2 + 2a - 3}$.

Simplify the following rational expressions.

14. $\frac{4 - k}{k^2 - 16}$

15. $\frac{m^2 - c^2}{c^2 - mc}$

16. $\frac{(k - 2)^2}{4 - k^2}$

17. $\frac{(m - 3)(m + 5)}{3(3 - m)(m + 5)}$

18. $\frac{m^2 + mk - 6k^2}{2k^2 + mk - m^2}$

19. $\frac{20k^3 + 26k^2 - 6k}{3 - 13k - 10k^2}$

✓ Independent Practice

1. Simplify $\frac{2k^2 - 3ck - 2c^2}{2c^2 + 3ck - 2k^2}$.

2. Simplify $\frac{(a + b)^2 - (m + c)^2}{a + b - m - c}$.

Reduce each of the following rational expressions to their lowest terms.

3. $\frac{16x^2 - v^2}{v^2 - 2xv - 24x^2}$

4. $\frac{m + c}{c - a}$

5. $\frac{3y^2 - 21y + 30}{18y^2 - 12y - 48}$

6. $\frac{4x^2 - 4x - 24}{10x^2 + 50x + 60}$

7. $\frac{k^3 + 10k^2 + 21k}{k^2 + 2k - 35}$

8. $\frac{2x^2 - 8y^2}{2x + 4y}$

9. $\frac{x^2 + 5xy - 14y^2}{x^2 + 4xy - 21y^2}$

10. $\frac{3a^2 - 12}{a^2 - 3a - 10}$

11. $\frac{6a^2 - 7ab - 5b^2}{6a^2 - 13ab + 5b^2}$

12. Matthew simplified $\frac{2x^2 - x - 21}{2x + 6}$ in this way:

$$\frac{\cancel{(x+3)}(2x-7)}{2\cancel{(x+3)}} = \frac{2x-7}{2} = x-7.$$

Explain the error that Matthew has made and then simplify the expression correctly.

Round Up

If you were to condense everything from this Section into a couple of points, they would be:

- **Rational expressions** are the same as fractions and are undefined if the denominator equals zero.
- A rational expression can be reduced to its lowest equivalent fraction by dividing out common factors of the numerator and denominator.

Topic 8.2.1

California Standards:

13.0: Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.

What it means for you:

You'll multiply rational expressions by factoring and canceling.

Key words:

- rational
- nonzero
- common factor

Section 8.2

Multiplying Rational Expressions

In Section 8.1 you learned about simplifying rational expressions. In this Topic you'll learn to multiply rational expressions, but then you'll use the same simplification methods to express your solutions in their simplest forms.

Multiplying by Rational Expressions

Given any nonzero expressions m , c , b , and v :

$$\frac{m}{c} \cdot \frac{b}{v} = \frac{m \cdot b}{c \cdot v} = \frac{mb}{cv}$$

That is, the product of two rational expressions is the **product of the numerators** divided by the **product of the denominators**.

These expressions can often be quite complicated, so simplify them as much as you can **before** multiplying.

- First, **factor** the numerators and denominators (if possible).
- Then find any **factors** that are **common** to both the top line (the numerator) and the bottom line (the denominator) and cancel them before multiplying.

Example 1

Simplify $\frac{ab}{2a+4b} \cdot \frac{5a+10b}{a^2b}$.

Solution

Step 1: Factor the numerators and denominators.

$$= \frac{a \cdot b}{2(a+2b)} \cdot \frac{5(a+2b)}{a \cdot a \cdot b}$$

Step 2: Cancel all the common factors and multiply.

$$\begin{aligned} &= \frac{\cancel{a}^1 \cdot \cancel{b}^1}{2(\cancel{a+2b})^1} \cdot \frac{5(\cancel{a+2b})^1}{\cancel{a}^1 \cdot a \cdot \cancel{b}^1} \\ &= \frac{5}{2a} \end{aligned}$$

Example 2

Multiply and simplify $\frac{a^2 - 4}{6 - a - a^2} \cdot \frac{2a + 6}{4a + 8}$.

Solution

Step 1: Factor the numerators and denominators if possible.

$$\begin{aligned} &= \frac{(a-2)(a+2)}{(a+3)(2-a)} \cdot \frac{2(a+3)}{2 \cdot 2(a+2)} \\ &= \frac{(a-2)(a+2)}{-1 \cdot (a-2)(a+3)} \cdot \frac{2(a+3)}{2 \cdot 2(a+2)} \end{aligned}$$

Step 2: Cancel all the common factors and multiply.

$$\begin{aligned} &= \frac{\cancel{(a-2)}^1 \cancel{(a+2)}^1}{-1 \cdot \cancel{(a-2)}^1 \cancel{(a+3)}^1} \cdot \frac{\cancel{2}^1 \cancel{(a+3)}^1}{\cancel{2}^1 \cdot 2 \cdot \cancel{(a+2)}^1} \\ &= -\frac{1}{2} \end{aligned}$$

Guided Practice

Multiply and simplify the rational expressions.

1. $\frac{14mc}{15ck} \cdot \frac{30kp}{28pt}$

2. $\frac{t-v}{p} \cdot \frac{tv}{5t-5v}$

3. $\frac{x+y}{x+2y} \cdot \frac{x+3}{x+y}$

4. $\frac{(p-2)(p+1)}{2(p+3)} \cdot \frac{p+3}{(p+1)(p+4)}$

5. $\frac{ab}{2a+4b} \cdot \frac{5a+10b}{a^2b}$

6. $\frac{3k^2-10k-8}{6k^2+7k+2} \cdot \frac{4k^2-4k-3}{3k^2-11k-4}$

You can extend this concept to the multiplication of **any number** of rational expressions, with **any number of variables**.

Example 3

Simplify $\frac{x^2 + 3x + 2}{-x - 3} \cdot \frac{14x - 2x^3}{x^2 - 1} \cdot \frac{4x^2 - x - 3}{16x^2 + 12x} \cdot \frac{x + 3}{x + 2}$.

Solution

Factor the numerators and denominators, and cancel all the common factors and multiply.

$$\begin{aligned} &= \frac{\cancel{(x+2)}^1 \cancel{(x+1)}^1}{-1 \cdot \cancel{(x+3)}^1} \cdot \frac{\cancel{2}^1 \cancel{x}^1 (7-x^2)}{\cancel{(x-1)}^1 \cancel{(x+1)}^1} \cdot \frac{\cancel{(4x+3)}^1 \cancel{(x-1)}^1}{2 \cdot \cancel{2}^1 \cancel{x}^1 (4x+3)} \cdot \frac{\cancel{(x+3)}^1}{\cancel{(x+2)}^1} \\ &= \frac{7-x^2}{-1 \cdot 2} = -\frac{7-x^2}{2} \text{ or } \frac{x^2-7}{2} \end{aligned}$$

Check it out:

Either of these answers is acceptable. The second is the simpler of the two forms though.

Example 4

Simplify $\frac{x^2 + 7x + 12}{2a^2 + 7a - 15} \cdot \frac{a^2 + 8a + 15}{ax - 3x + 4a - 12} \cdot \frac{x^2 + 9x + 18}{x^2 + 6x + 9}$.

Solution

Factor the numerators and denominators, then cancel all the common factors and multiply.

$$\begin{aligned} &= \frac{\cancel{(x+3)}\cancel{(x+4)}_1 \cdot \cancel{(a+5)}(a+3)_1 \cdot (x+6)\cancel{(x+3)}_1}{(2a-3)\cancel{(a+5)}_1 \cdot (a-3)\cancel{(x+4)}_1 \cdot \cancel{(x+3)}\cancel{(x+3)}_1} \\ &= \frac{(a+3)(x+6)}{(2a-3)(a-3)} \quad \text{or} \quad \frac{ax+3x+6a+18}{2a^2-9a+9} \end{aligned}$$

Check it out:

You can leave your answer in either “factored” or “multiplied out” form — they’re both equally valid.

✓ Guided Practice

Multiply and simplify the rational expressions.

7. $\frac{x^2 - 1}{x^2 + x - 2} \cdot \frac{x^2 + 5x + 6}{x^2 - 3x - 4} \cdot \frac{x^2 - 9x + 20}{x + 3}$

8. $\frac{2 + m - m^2}{m^2 - 4} \cdot \frac{m^2 - m - 6}{-2m + 6} \cdot \frac{-4m + 12}{-m + 3}$

9. $\frac{x^2 + 3xy + 2y^2}{x^2 + xy - 2y^2} \cdot \frac{x^2 - y^2}{x^2 - xy - 2y^2} \cdot \frac{(x - 2y)^2}{(x + y)^2}$

✓ Independent Practice

Multiply and simplify the rational expressions.

1. $\frac{2x^2 + x - 1}{x^2 + 2x - 3} \cdot \frac{x^2 - 6x + 5}{x^2 - 4x - 5}$

2. $\frac{a^2 - b^2}{b^2 + ab - 2a^2} \cdot \frac{b^2 + 5ab + 6a^2}{2b^2 + 7ab + 3a^2}$

3. $\frac{x^2 - 2x - 3}{3 - 4x + x^2} \cdot \frac{2 - 3x + x^2}{x^2 - x - 2}$

4. $\frac{a^2 - k^2}{a^2 + 3ak + 2k^2} \cdot \frac{a^2 + 5ak + 6k^2}{-2a + 2k}$

5. $\frac{a^2 + ab - 2b^2}{b^2 - 3ab + 2a^2} \cdot \frac{b^2 + ab - 6a^2}{2a^2 + 3ab - 2b^2} \cdot \frac{a^2 - b^2}{2a + 2b}$

6. $\frac{3x^2 + 27x + 60}{x(x^2 + 6x + 8)} \cdot \frac{x(x^2 + 5x + 6)}{2(x^2 + 7x + 12)} \cdot \frac{(x+1)(x+4)}{(x+3)(x+5)}$

7. $\frac{6x^2 + 6x - 36}{3x^3 - 15x^2 + 18x} \cdot \frac{x^3 + 9x^2 + 20x}{2x^2 - 8x - 42} \cdot \frac{x^2 - 9}{x^2 + 10x + 24}$

Round Up

Usually the most difficult thing when solving problems like these is factoring the numerators and denominators. Look for “difference of two squares” expressions, perfect squares, and minus signs that you can factor outside the parentheses.

Dividing Rational Expressions

California Standards:

13.0: Students add, subtract, multiply, and **divide rational expressions and functions**. Students solve both computationally and conceptually challenging problems by using these techniques.

What it means for you:

You'll divide rational expressions by factoring and canceling.

Key words:

- rational
- reciprocal
- common factor

Don't forget:

See Topic 1.2.2 for more on reciprocals.

Check it out:

The middle numbers in these two columns are reciprocals of each other.

Dividing by rational expressions is a lot like multiplying — you just have to do an extra step first.

Dividing is the Same as Multiplying by the Reciprocal

Given any **nonzero expressions** m , c , b , and v :

$$\frac{m}{c} \div \frac{b}{v} = \frac{m}{c} \cdot \frac{v}{b} = \frac{mv}{cb}$$

That is, to divide $\frac{m}{c}$ by $\frac{b}{v}$, **multiply** by the **reciprocal** of $\frac{b}{v}$.

You can extend this concept to the division of **any** rational expression.

Suppose you pick a number such as 10 and divide by, say, $\frac{1}{2}$.

The question you're trying to answer is "How many times does $\frac{1}{2}$ go into 10?" or "How many halves are in 10?"

<u>Division</u>	<u>Equivalent to</u>
$10 \div \frac{1}{2} = 20$	$10 \times 2 = 20$
$10 \div \frac{1}{3} = 30$	$10 \times 3 = 30$
$10 \div \frac{1}{4} = 40$	$10 \times 4 = 40$
$10 \div \frac{1}{n} = 10n$	$10 \times n = 10n$

So, 10 divided by a fraction is equivalent to 10 multiplied by the reciprocal of that fraction. Dividing **anything** by a rational expression is the same as multiplying by the reciprocal of that expression.

So you can **always** rewrite an expression $a \div b$ in the form $a \cdot \frac{1}{b} = \frac{a}{b}$ (where b is any nonzero expression).

You Should Still Cancel Common Factors

Example 1

Simplify $\frac{k^2 - 25}{2k} \div (k + 5)$.

Solution

$\frac{k^2 - 25}{2k} \div (k + 5)$ can be written as $\frac{k^2 - 25}{2k} \div \frac{k + 5}{1}$

Rewrite the division as a multiplication by the reciprocal of the divisor.

$$= \frac{k^2 - 25}{2k} \cdot \frac{1}{k + 5}$$

Factor as much as you can.

$$= \frac{(k - 5)(k + 5)}{2k} \cdot \frac{1}{(k + 5)}$$

Cancel any common factors between the numerators and denominators.

$$= \frac{(k - 5)\cancel{(k + 5)}^1}{2k} \cdot \frac{1}{\cancel{(k + 5)}^1} = \frac{k - 5}{2k}$$

Check your answer. Multiply your answer by $(k + 5)$:

$$\frac{k - 5}{2k} \cdot \frac{k + 5}{1} = \frac{(k - 5)(k + 5)}{2k} = \frac{k^2 - 25}{2k} \quad \checkmark$$

Example 2

Simplify $\frac{m^2 - 4}{m^2 - 3m + 2} \div \frac{2m}{m - 1}$.

Solution

Rewrite the division as a multiplication by the reciprocal of the divisor.

$$= \frac{m^2 - 4}{m^2 - 3m + 2} \cdot \frac{m - 1}{2m}$$

Factor all numerators and denominators.

$$= \frac{(m + 2)(m - 2)}{(m - 2)(m - 1)} \cdot \frac{m - 1}{2m}$$

Cancel any common factors between the numerators and denominators.

$$= \frac{(m + 2)\cancel{(m - 2)}^1 \cdot \cancel{m - 1}^1}{\cancel{(m - 2)}^1 \cdot \cancel{(m - 1)}^1} \cdot \frac{1}{2m} = \frac{m + 2}{2m}$$

✓ Guided Practice

Divide and simplify each expression.

$$1. \frac{b^2c^2d^2}{bcd^3} \div \frac{bcd^2}{abc}$$

$$2. \frac{a(b-2)}{b+1} \div \frac{2a}{(b+1)(b-1)}$$

$$3. \frac{a^2-9}{a^2+a-6} \div \frac{a+3}{a-2}$$

$$4. \frac{a^2+3a+2}{a^2-a-6} \div \frac{a^2-1}{a^2-4a+3}$$

$$5. \frac{x^2-5x-6}{x^2+3x-10} \div \frac{x^2-4x-5}{x^2-25}$$

Check it out:

It makes a difference which order you do the divisions because division is not associative, which means: $(a \div b) \div c \neq a \div (b \div c)$

Check it out:

In practice, this means that you can rewrite each division as a multiplication by the reciprocal of the divisor.

You Can Divide Long Strings of Expressions At Once

Just like multiplication, you can divide **any number** of rational expressions at once, but it makes a big difference which **order** you do things in.

If there are no parentheses, you always work through the calculation **from left to right**, so that:

$$\frac{a}{b} \div \frac{c}{d} \div \frac{e}{f} = \frac{a}{b} \cdot \frac{d}{c} \div \frac{e}{f} = \frac{a}{b} \cdot \frac{d}{c} \cdot \frac{f}{e}$$

Example 3

Simplify $\frac{x^2+5x+6}{x^2+3x} \div \frac{x^2+x-2}{2x^2+2x} \div \frac{x^2+2x+1}{x-1}$.

Solution

Rewrite each division as a multiplication by the reciprocal of the divisor.

$$\begin{aligned} &= \frac{x^2+5x+6}{x^2+3x} \cdot \frac{2x^2+2x}{x^2+x-2} \cdot \frac{x^2+2x+1}{x-1} \\ &= \frac{x^2+5x+6}{x^2+3x} \cdot \frac{2x^2+2x}{x^2+x-2} \cdot \frac{x-1}{x^2+2x+1} \end{aligned}$$

Factor all numerators and denominators.

$$= \frac{(x+2)(x+3)}{x(x+3)} \cdot \frac{2x(x+1)}{(x+2)(x-1)} \cdot \frac{x-1}{(x+1)(x+1)}$$

Cancel any common factors between the numerators and denominators.

$$\begin{aligned} &= \frac{\cancel{(x+2)}\cancel{(x+3)}_1}{\cancel{x}\cancel{(x+3)}_1} \cdot \frac{2\cancel{x}\cancel{(x+1)}_1}{\cancel{(x+2)}\cancel{(x-1)}_1} \cdot \frac{\cancel{x-1}_1}{\cancel{(x+1)}\cancel{(x+1)}} \\ &= \frac{2}{x+1} \end{aligned}$$

Parentheses override this order of operations, so you need to simplify any expressions in parentheses first:

$$\frac{a}{b} \div \left(\frac{c}{d} \div \frac{e}{f} \right) = \frac{a}{b} \div \frac{c \cdot f}{d \cdot e} = \frac{a}{b} \cdot \frac{d \cdot e}{c \cdot f}$$

✓ Guided Practice

Divide and simplify each expression.

6. $\frac{k^2 - 1}{2k^2 - 14k} \div \frac{k^2 + 5k - 6}{k^2 - 9k + 14} \div \frac{-k^2 + 3k - 2}{2k^2 - 10k}$

7. $\frac{x^2 - 4x - 12}{2x^2 - 3x - 2} \div \frac{-x^2 + 2x + 8}{3x^3 + 3x^2 - 18x} \div \frac{6x^3 - 36x^2}{-2x^2 + 7x + 4}$

8. $\frac{(x-2)}{(x+3)} \div \left(\frac{(x-2)(x+4)}{(x+3)} \div \frac{(x+4)}{(x+1)} \right)$

You Can Multiply and Divide at the Same Time

Say you have an expression like this to simplify: $\frac{a}{b} \div \frac{c}{d} \times \frac{e}{f}$

Again, you work from left to right, and anywhere you get a **division**, multiply by the reciprocal, so:

$$\frac{a}{b} \div \frac{c}{d} \times \frac{e}{f} = \frac{a}{b} \times \frac{d}{c} \times \frac{e}{f}$$

Example 4

Simplify $\frac{p^2 + pq - 2q^2}{p^2 - 2pq - 3q^2} \times \frac{p^2 - q^2}{pq + 2q^2} \div \frac{p^2 - 2pq + q^2}{p^2 - 3pq}$.

Solution

Rewrite any divisions as multiplications by reciprocals.

$$= \frac{p^2 + pq - 2q^2}{p^2 - 2pq - 3q^2} \times \frac{p^2 - q^2}{pq + 2q^2} \times \frac{p^2 - 3pq}{p^2 - 2pq + q^2}$$

Factor all numerators and denominators.

$$= \frac{(p+2q)(p-q)}{(p-3q)(p+q)} \times \frac{(p-q)(p+q)}{q(p+2q)} \times \frac{p(p-3q)}{(p-q)(p-q)}$$

Cancel any common factors between the numerators and denominators.

$$= \frac{\cancel{(p+2q)}\cancel{(p-q)}^1}{\cancel{(p-3q)}\cancel{(p+q)}^1} \times \frac{\cancel{(p-q)}^1\cancel{(p+q)}^1}{q\cancel{(p+2q)}^1} \times \frac{p\cancel{(p-3q)}^1}{\cancel{(p-q)}^1\cancel{(p-q)}^1}$$

$$= \frac{p}{q}$$

Make Sure You Can Justify All Your Steps

Example 5

Show that $\frac{2a^2 - 7a + 3}{a^2 + 4a - 21} \div \frac{2a^2t + at - t}{2a^2t + 12at - 14t} = \frac{2a - 2}{a + 1}$. Justify your work.

Solution

$\frac{2a^2 - 7a + 3}{a^2 + 4a - 21} \div \frac{2a^2t + at - t}{2a^2t + 12at - 14t}$	Start with left-hand side
$= \frac{2a^2 - 7a + 3}{a^2 + 4a - 21} \times \frac{2a^2t + 12at - 14t}{2a^2t + at - t}$	Definition of division
$= \frac{(2a - 1)(a - 3)}{(a + 7)(a - 3)} \times \frac{2t(a^2 + 6a - 7)}{t(2a^2 + a - 1)}$	Distributive property
$= \frac{(2a - 1)(a - 3)}{(a + 7)(a - 3)} \times \frac{2t(a + 7)(a - 1)}{t(2a - 1)(a + 1)}$	Distributive property
$= \frac{2(a - 1)}{a + 1} \times \frac{t(2a - 1)(a + 7)(a - 3)}{t(2a - 1)(a + 7)(a - 3)}$	Commutative and associative properties of multiplication
$= \frac{2(a - 1)}{a + 1}$	Inverse and identity properties
$= \frac{2a - 2}{a + 1}$	Distributive property

✔ Guided Practice

Simplify these rational expressions.

9. $\frac{t^2 - 1}{t^2 + 2t - 3} \div \frac{t + 1}{t^2 + 4t + 3} \times \frac{t - 1}{1}$

10. $\frac{a^2 + a - 12}{a^2 + a - 2} \div \frac{a^2 + 5a + 4}{a^2 + 2a + 1} \times \frac{a^2 + 2a - 3}{a^2 - 2a - 3}$

11. $\frac{x^2 + 5x - 14}{x^2 - 4x - 21} \div \frac{x^2 + 6x - 7}{x^2 - 6x - 7} \times \frac{x^2 + 2x - 3}{x^2 - 5x + 6}$

12. $\frac{a^2 - 1}{a^2 - 4} \div \frac{a^2 - 2a - 3}{a^2 - 3a - 10} \times \frac{a^2 - 5a + 6}{-a^2 + 2a + 15}$

✓ Independent Practice

Divide and simplify each expression.

$$1. \frac{k^2 - m^2}{2k^2 + km - m^2} \div \frac{2k + 2m}{2k^2 + 3km - 2m^2}$$

$$2. \frac{t^2 + 2t - 3}{t^2 + 4t + 3} \div \frac{3t - 3}{t^2 - t - 2}$$

$$3. \frac{-x^3 - 3x^2 - 2x}{x^2 - 2x - 3} \div \frac{x^2 - x - 6}{x^3 - 2x^2 - 3x}$$

$$4. \frac{b^3 - 4b}{b^3 + b} \div \frac{b^2 - b - 2}{b^4 - 1}$$

$$5. \frac{x^2 - 6x + 8}{x^2 - 4} \div \frac{-x^3 + x}{-2x^2 + 4x + 16}$$

$$6. \frac{y^2 - y - 2}{y^2 + 3y - 4} \div \frac{-y + 2}{y^2 - 3y + 2}$$

$$7. \frac{a^3 - 4a}{-a^2 + 2a} \div \frac{a^2 + a - 2}{a^2 - a - 2}$$

$$8. \frac{b^2 - 1}{b^2 - 2b - 3} \div \frac{b^2 - 2b + 1}{b^2 - 4b + 3}$$

$$9. \frac{(m - v)^2}{m^2 - v^2} \div \frac{m^2 - 3mv + 2v^2}{(m - 2v)^2}$$

$$10. \frac{x^2 - 3x + 2}{x^2 + x - 2} \div \frac{-x + 2}{x^2 - 3x - 10}$$

$$11. \frac{2x^2 - 5x - 12}{4x^2 + 8x + 3} \div \frac{x^2 - 16}{2x^2 + 7x + 3} \div \frac{x^2 - 9}{x^2 + 2x - 8}$$

$$12. \frac{y + 5}{y^2 - 4y - 5} \div \frac{y^2 + 4y - 5}{y + 1} \div \frac{1}{y^2 - 6y + 5}$$

$$13. \frac{t^2 + t - 6}{t^2 + 6t + 9} \div (t^2 - 4) \div \frac{t + 2}{t + 3}$$

Simplify these rational expressions.

$$14. \frac{k^2 - 5k + 6}{k^2 + 2k - 8} \cdot \frac{-2k^2 - 6k - 4}{k^2 - 2k - 3} \div \frac{k^2 + 3k + 2}{k^2 + 5k + 4}$$

$$15. \frac{-2v^2 + 4vw}{3v^2 - 4vw + w^2} \times \frac{v^2 - w^2}{-2vw + 3w^2} \div \frac{v^3 - vw^2}{6v^2w + 4vw^2 - 2w^3} \times \frac{-2v^2 + 5vw - 3w^2}{-4v^2 + 4vw + 8w^2}$$

$$16. \frac{m^2 + 2mn + n^2}{m^2n - 3mn^2} \div \frac{4m^2 + 5mn + n^2}{2m^2 - 5mn - 3n^2} \div \frac{2m^2 + 3mn + n^2}{-m^2 + 3mn - 2n^2} \div \frac{-m^2 + 3mn - 2n^2}{-2m^2n + 5mn^2}$$

$$17. \frac{m^2 + 2mn + n^2}{m^2n - 3mn^2} \div \left(\frac{4m^2 + 5mn + n^2}{2m^2 - 5mn - 3n^2} \div \frac{2m^2 + 3mn + n^2}{-m^2 + 3mn - 2n^2} \right) \div \frac{-m^2 + 3mn - 2n^2}{-2m^2n + 5mn^2}$$

Round Up

It's really important that you can *justify your work step by step*, because division of rational expressions can involve lots of calculations that look quite similar.

Topic 8.3.1

California Standards:

13.0: Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.

What it means for you:

You'll add and subtract rational numbers with the same denominators.

Key words:

- common denominator
- rational
- common factor

Section 8.3

Fractions with Identical Denominators

In Section 8.2 you dealt with multiplying and dividing rational expressions, so you can probably guess that *adding and subtracting* is next. First up are fractions with the *same denominators*, which are the easiest kind to add and subtract.

Common Denominators Make Things Easier

To **add** or **subtract** two rational expressions with the **same denominator**, just add or subtract the **numerators**, then put the result over the common denominator. Here's a simple numerical example:

$$\frac{5}{9} + \frac{2}{9} = \frac{5+2}{9} = \frac{7}{9}$$

Add the numerators and divide the answer by the common denominator.

$$\frac{5}{9} - \frac{2}{9} = \frac{5-2}{9} = \frac{3}{9} = \frac{1}{3}$$

Subtract the numerators and divide the answer by the common denominator.

Reduce your answer to its simplest form.

In general, given any **expressions** m , c , and v , where v is **nonzero**:

$$\frac{m}{v} + \frac{c}{v} = \frac{m+c}{v} \quad \text{and} \quad \frac{m}{v} - \frac{c}{v} = \frac{m-c}{v}$$

Example 1

Simplify $\frac{k+1}{t} + \frac{2k-5}{t}$.

Solution

Add the numerators and divide by the common denominator:

$$\begin{aligned} &= \frac{(k+1) + (2k-5)}{t} \\ &= \frac{k+2k+1-5}{t} \\ &= \frac{3k-4}{t} \end{aligned}$$

Example 2

Simplify $\frac{k+1}{t} - \frac{2k-5}{t}$, justifying each step.

Solution

Subtract the numerators:

$$= \frac{(k+1) - (2k-5)}{t}$$

Divide the difference by the common denominator, t

$$= \frac{k+1-2k+5}{t}$$

Use distributive property to simplify $-(2k-5)$

$$= \frac{k-2k+1+5}{t}$$

Commutative and associative properties of addition

$$= \frac{-k+6}{t}$$

Combine like terms

✓ Guided Practice

Simplify each expression.

1. $\frac{2x-5}{x} + \frac{3x+4}{x}$

2. $\frac{5}{2} - \frac{x-3}{2}$

3. $\frac{-5k-3}{2k-1} + \frac{3k+2}{2k-1}$

4. $\frac{x^2+5x+4}{x^2-5x+4} - \frac{x^2+4x+3}{x^2-5x+4}$

5. $\frac{2x^2-5x+3}{2x^2+5x+3} - \frac{2x^2+x-6}{2x^2+5x+3}$

6. $\frac{b^2-5b+4}{7b} + \frac{2b^2+5b+3}{7b} + \frac{2b+7}{7b}$

Cancel Any Common Factors**Example 3**

Add and simplify $\frac{5m}{4} + \frac{7m-4}{4}$.

Solution

Add the numerators and divide by the common denominator.

$$= \frac{5m+7m-4}{4}$$

$$= \frac{12m-4}{4}$$

$$= \frac{\cancel{4}(3m-1)}{\cancel{4}}$$

Factor the numerator and cancel any common factors

$$= 3m-1$$

Example 4

Subtract and simplify $\frac{3a+1}{a^2-9} - \frac{2a+4}{a^2-9}$.

Solution

Subtract the numerators and divide by the common denominator.

$$\begin{aligned} &= \frac{(3a+1) - (2a+4)}{a^2-9} \\ &= \frac{3a+1-2a-4}{a^2-9} \\ &= \frac{a-3}{a^2-9} \\ &= \frac{\cancel{a}^1 \cancel{-3}^1}{\cancel{a}^1(a+3)} \\ &= \frac{1}{a+3} \end{aligned}$$

Guided Practice

Simplify each expression.

$$7. \frac{-2a-4}{a+1} - \frac{-4a-6}{a+1}$$

$$8. \frac{4x-4}{x-1} + \frac{3x-3}{x-1} + \frac{x^2+x-2}{x-1}$$

$$9. \frac{3x+12}{x+4} + \frac{x^2+5x+4}{x+4}$$

$$10. \frac{4b^2-5b+2}{9b^2-4} + \frac{2b^2-3b+1}{9b^2-4} + \frac{3b^2-4b+1}{9b^2-4}$$

Independent Practice

Simplify each expression.

$$1. \frac{x^2+6x+8}{x^2+4x+3} + \frac{x^2+11x+28}{x^2+4x+3}$$

$$2. \frac{2x^2+7x+3}{2x^2+3x-9} - \frac{x^2+2x-35}{2x^2+3x-9}$$

$$3. \frac{x^2-4x-5}{2x^2+9x+4} - \frac{2x^2+13x+18}{2x^2+9x+4}$$

$$4. \frac{3(b+3)}{4b} + \frac{b-5}{4b}$$

$$5. \frac{2c^2+c+17}{c^2+7c+12} - \frac{c^2-5c+9}{c^2+7c+12}$$

$$6. \left(\frac{2x-1}{4x^2-1} \right) \left(\frac{x^2-2x-3}{2x^2-9x+4} - \frac{5x-x^2+1}{2x^2-9x+4} \right)$$

$$7. \left(\frac{2x^2+7x+5}{x^2-4} \right) \left(\frac{x(2x-9)-10}{6x^2+23x+20} + \frac{x^2+7x+2}{6x^2+23x+20} \right)$$

Round Up

When subtracting algebraic fractions, be sure to *distribute the minus sign* over the whole parenthesis that follows it, *changing any plus signs to minus signs and vice versa*. And make sure you reduce your answer to its *simplest form* — look for “*difference of two squares*” patterns.

Fractions with Different Denominators

California Standards:

13.0: Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.

What it means for you:

You'll add and subtract rational numbers with different denominators.

Key words:

- common denominator
- rational
- common factor
- least common multiple

Don't forget:

See Topic 1.3.4 for more on equivalent fractions.

*For you to be able to add or subtract two rational expressions, they must have a **common denominator**.*

*If the two fractions you start with have different denominators, you need to find equivalent fractions with a common denominator **before** you can add or subtract them.*

You Need to Find a Common Denominator

In general, you can find a common denominator by simply **multiplying** the denominators together. You then convert each fraction to an **equivalent** fraction with this common denominator.

Example 1

Add $\frac{2}{3} + \frac{1}{5}$.

Solution

Multiply the denominators to get a common denominator (3·5). Convert each fraction into an equivalent fraction with the common denominator and respective numerators 2·5 and 1·3.

$$\frac{2 \cdot 5}{3 \cdot 5} + \frac{1 \cdot 3}{5 \cdot 5} = \frac{10}{15} + \frac{3}{15}$$

Once you have two fractions with the same denominator, add the numerators and divide by the common denominator.

$$= \frac{13}{15}$$

So — given any **expressions** m , c , b , and v , where c and v are **nonzero**:

$$\frac{m}{c} + \frac{b}{v} = \frac{mv}{cv} + \frac{bc}{cv} = \frac{mv + bc}{cv}$$

and

$$\frac{m}{c} - \frac{b}{v} = \frac{mv}{cv} - \frac{bc}{cv} = \frac{mv - bc}{cv}$$

Example 2

Subtract $\frac{4m-1}{4} - \frac{3m+5}{5}$.

Solution

Find a common denominator: $4 \times 5 = 20$

Convert each fraction to an equivalent fraction with 20 as the denominator.

$$\frac{4m-1}{4} = \frac{5(4m-1)}{4 \cdot 5} = \frac{20m-5}{20} \quad \text{and} \quad \frac{3m+5}{5} = \frac{4(3m+5)}{4 \cdot 5} = \frac{12m+20}{20}$$

Subtract the equivalent fractions.

$$\begin{aligned} \frac{20m-5}{20} - \frac{12m+20}{20} &= \frac{(20m-5) - (12m+20)}{20} \\ &= \frac{20m-5-12m-20}{20} \\ &= \frac{8m-25}{20} \end{aligned}$$

Distribute the minus sign and simplify

Don't forget:

See Topic 1.3.6 for more on the LCM.

In Examples 1 and 2, the LCM of the denominators is simply their product.

This method works well with fairly simple examples like these, but when you have more complicated problems to deal with, it can become a lengthy process. A better alternative is to find the **least** (or lowest) **common multiple** — the **LCM**.

Guided Practice

Use the method of multiplying denominators to simplify the following.

1. $\frac{x+1}{2} + \frac{x+9}{8}$

2. $\frac{x^2-2x+1}{6} - \frac{x^2+4x+4}{4}$

3. $\frac{x^2+6x+8}{4} + \frac{x^2+5x+6}{5}$

4. $\frac{a^2-1}{6} - \frac{4a^2-16}{12}$

Simplify each expression by first finding the LCM of the denominators.

5. $\frac{c^2-5c-24}{8} + \frac{8c^2+2c-3}{12}$

6. $\frac{3y^2+17y+20}{10} - \frac{8y^2-2y-3}{5}$

7. $\frac{2b^2+3b-9}{2} - \frac{3b^2+4b-4}{3} + \frac{4b^2+3b-1}{8}$

8. $\frac{y^2-1}{4} - \frac{y^2-4}{5} - \frac{y^2-9}{10}$

9. $\frac{2x^2+x+1}{15} + \frac{4x^2+8x+3}{2} + \frac{8x^2+14x+3}{10} - \frac{6x^2+11x+4}{5}$

Finding the Least Common Multiple (LCM)

The least common multiple of two or more denominators is the **smallest possible number** (or simplest expression) that's divisible by **all** denominators.

To find the LCM:

- **Factor** the denominators **completely**.
- Multiply together the **highest power of each factor** that appears in any of the denominators.
- Any factor that appears in more than one denominator should only be included once in the LCM.

Using the LCM as your common denominator makes the problem as simple as possible.

Adding or Subtracting Fractions with Different Denominators

When adding or subtracting fractions:

- 1) Find the least common multiple of the denominators.
- 2) Convert each fraction into an equivalent fraction with the LCM as its denominator.
- 3) Add or subtract the numerators and divide by the common denominator (LCM).
- 4) Factor both the numerator and denominator if possible, and cancel any common factors.

Example 3

Simplify $\frac{2-y}{x^2y(x+1)} + \frac{5+x}{x(x+1)^2}$.

Solution

Step 1: Find the LCM of $x^2y(x+1)$ and $x(x+1)^2$:

$$\text{LCM} = x^2y(x+1)^2$$

Step 2: Convert each fraction to an equivalent fraction with $x^2y(x+1)^2$ as the denominator.

$$\frac{2-y}{x^2y(x+1)} = \frac{(2-y)(x+1)}{x^2y(x+1) \cdot (x+1)} = \frac{(2-y)(x+1)}{x^2y(x+1)^2}$$

$$\frac{5+x}{x(x+1)^2} = \frac{xy(5+x)}{xy \cdot x(x+1)^2} = \frac{xy(5+x)}{x^2y(x+1)^2}$$

Check it out:

The highest power of x is x^2 .
The highest power of y is y .
The highest power of $(x+1)$ is $(x+1)^2$.
So the LCM is $x^2y(x+1)^2$.

Example 3 continued

Step 3: Add the equivalent fractions.

$$\begin{aligned} & \frac{(2-y)(x+1)}{x^2y(x+1)^2} + \frac{xy(5+x)}{x^2y(x+1)^2} \\ &= \frac{(2-y)(x+1) + xy(5+x)}{x^2y(x+1)^2} \\ &= \frac{2x - xy - y + 2 + xy(5+x)}{x^2y(x+1)^2} \end{aligned}$$

✓ Guided Practice

Simplify each expression.

10. $\frac{4}{2k-6} + \frac{3}{12-4k}$

11. $\frac{a-1}{k+1} - \frac{a-1}{(k+1)^2}$

12. $\frac{k+7}{k-6} + \frac{k-6}{k+7}$

13. $\frac{x+1}{x-1} - \frac{2}{x+1}$

Add Lots of Fractions Using a Common Denominator

If you want to add or subtract more than two fractions at once, you have to put **all** the numerators over a common denominator.

Example 4

Simplify $\frac{5}{2x+6} + \frac{4}{4x-4} - \frac{5(x+1)}{x^2+2x-3}$.

Solution

Step 1: Find the LCM of $2x+6$, $4x-4$, and x^2+2x-3 .

Factor each denominator:

$$2x+6 = 2 \cdot (x+3)$$

$$4x-4 = 4 \cdot (x-1) = 2^2 \cdot (x-1)$$

$$x^2+2x-3 = (x-1)(x+3)$$

$$\text{LCM} = 4(x-1)(x+3)$$

Check it out:

The highest power of 2 is $2^2 = 4$.

The highest power of $(x-1)$ is $(x-1)$.

Highest power of $(x+3)$ is $(x+3)$.

So the LCM is $4(x-1)(x+3)$.

Example 4 continued

Step 2: Convert each fraction to an equivalent fraction with $4(x-1)(x+3)$ as the denominator.

$$\frac{5}{2x+6} = \frac{5}{2(x+3)} = \frac{5 \cdot 2(x-1)}{2(x+3) \cdot 2(x-1)} = \frac{10(x-1)}{4(x-1)(x+3)}$$

$$\frac{4}{4x-4} = \frac{4}{4(x-1)} = \frac{4(x+3)}{4(x-1)(x+3)}$$

$$\frac{5(x+1)}{x^2+2x-3} = \frac{5(x+1)}{(x-1)(x+3)} = \frac{4 \cdot 5(x+1)}{4(x-1)(x+3)} = \frac{20(x+1)}{4(x-1)(x+3)}$$

Step 3: Add or subtract the equivalent fractions.

$$\begin{aligned} & \frac{10(x-1)}{4(x-1)(x+3)} + \frac{4(x+3)}{4(x-1)(x+3)} - \frac{20(x+1)}{4(x-1)(x+3)} \\ &= \frac{10(x-1) + 4(x+3) - 20(x+1)}{4(x-1)(x+3)} \\ &= \frac{10x - 10 + 4x + 12 - 20x - 20}{4(x-1)(x+3)} \\ &= \frac{-6x - 18}{4(x-1)(x+3)} \\ &= \frac{-6(x+3)}{4(x-1)(x+3)} \\ &= \frac{\cancel{2} \cdot 3 \cdot \cancel{(x+3)}^1}{\cancel{2} \cdot 2(x-1) \cdot \cancel{(x+3)}^1} \\ &= -\frac{3}{2(x-1)} \end{aligned}$$

Check it out:

Simplify your answer as much as possible — factor the numerator and denominator and cancel any common factors.

Guided Practice

Simplify each expression.

14. $\frac{1}{m+1} + \frac{2}{m-2} + \frac{3}{m+3}$

15. $\frac{1}{a-1} + \frac{2}{a-2} - \frac{2a-3}{a^2-3a+2}$

16. $\frac{1}{x} - \frac{x}{x-3} - 1$

17. $\frac{1}{2k^2+k-1} + \frac{2}{2k^2-3k+1}$

18. $\frac{y^2+3y}{y^2+y} + \frac{y+9}{y^2+2y} - \frac{y^2-1}{y^2+3y+2}$

19. $\frac{a-1}{a^2+5a} - \frac{a-1}{a^3+5a^2} + \frac{1}{a^3+6a^2+5a} + 1$

Independent Practice

Simplify each expression.

$$1. \frac{x+3}{6} + \frac{x^2-x-2}{8}$$

$$2. \frac{x^2+4x+4}{3} - \frac{x^2+4x-16}{5}$$

$$3. \frac{a^2+3a+2}{2} - \frac{2a^2+6a+8}{3} + \frac{2a^2+4a+10}{4} - \frac{a^2-a+4}{5}$$

$$4. \frac{a^2+a+4}{4} + \frac{a^2+3a+11}{5} - \frac{a^2-3a+7}{2} + \frac{a^2+a+4}{10}$$

$$5. \frac{ab+2a-4}{b} + \frac{4-a}{a}$$

$$6. \frac{2y^2-xy+1}{x} - \frac{y-2xy+1}{y}$$

$$7. \frac{m+c}{c} - \frac{m-c}{m}$$

$$8. \frac{4x}{5x+15} - \frac{x-1}{x+3}$$

$$9. \frac{1}{t^2-1} - \frac{1}{(t-1)^2}$$

$$10. \frac{3}{y^2+7y+12} - \frac{2}{y^2+3y-4}$$

$$11. \frac{a}{a-1} + \frac{2a}{a+1} - \frac{3a}{a-1}$$

$$12. \frac{1}{x-3} - \frac{2}{x+3} + \frac{3}{x-3}$$

$$13. \frac{w^2-1}{w} - \frac{x^2-1}{x} - \frac{y^2-1}{y}$$

$$14. \frac{c}{3c+2} + \frac{2c}{3c^2-4c-4} + \frac{c+3}{2c-4}$$

$$15. \frac{2y}{x^2+xy} - \frac{2y}{2x^2+3xy+y^2} + \frac{1}{x}$$

$$16. \frac{k+3}{k^2-3k+2} + \frac{7k+5}{k^2+k-2}$$

$$17. \frac{1}{2-x} - \frac{3x}{x^2-4}$$

$$18. \frac{x-1}{x^2+3x+2} - \frac{2x-1}{x^2+x-2}$$

$$19. \frac{2p+q}{q+1} + \frac{2-p^2}{p(q-1)} + \frac{q^2-q}{1-q^2}$$

$$20. \frac{2-k}{1-2k} - \frac{2+k}{1+2k} - \frac{1-6k}{4k^2-1}$$

$$21. \frac{a}{a^2+5a-6} - \frac{a}{a+6} \times \frac{1}{a-1}$$

$$22. \frac{3}{y-4} \times \frac{y-1}{y+2} + \frac{y-2}{y+2} \times \frac{3}{y-4}$$

Round Up

Remember — you can only add or subtract fractions when they've got the **same denominators**. That means that the first step when you're dealing with fractions with different denominators is to find a **common denominator**. Then you can use the same methods you saw in Topic 8.3.1.

Topic 8.4.1

Section 8.4

Solving Fractional Equations

California Standards:

13.0: Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.

What it means for you:

You'll solve equations involving rational numbers.

Key words:

- common denominator
- rational
- common factor
- limitation
- undefined
- least common multiple

*This Topic uses all the skills that you've learned in the earlier Topics in this Section — but now you'll be dealing with **equations** rather than just **expressions**.*

There are Limitations on Fraction Equation Solutions

A fractional equation is **undefined** when the denominator of any of its expressions equals **zero**. This means that there are **limitations** on the possible **solutions** of a fractional **equation**.

For example, for the fractional equation $\frac{1}{3x-1} = \frac{2}{x+1}$ the **limitations** on x are that $3x - 1 \neq 0$ and $x + 1 \neq 0$, that is $x \neq \frac{1}{3}$ and $x \neq -1$.

✓ Guided Practice

State the limitations on the possible solutions of these equations.

1. $\frac{4}{x-1} = \frac{3}{x+9}$

2. $\frac{7}{2x+9} + \frac{3}{x-3} = \frac{4}{2x-1}$

3. $\frac{5}{x^2-x-6} = \frac{1}{x}$

4. $\frac{x}{2x^2-5x-3} = \frac{x+1}{4x^2-1}$

5. $\frac{x+4}{x^2+2x-3} + \frac{x^2+9x+13}{x^2+x-2} = \frac{x^2-9}{x^2-4}$

Solve Using the LCM of the Denominators

When solving fractional equations where the **rational expressions** have **different denominators**, multiply the entire equation by the **least common multiple** (LCM) of the denominators.

This allows all the denominators to be canceled out, which makes the equation easier to solve.

Example 1

Solve the fractional equation $\frac{3}{8} + \frac{1}{3x-1} = \frac{2}{x+1}$.

Solution

To solve for x , find the least common multiple of the denominators.

In this example, the LCM of the denominators is simply the denominators multiplied together, i.e. $8(3x-1)(x+1)$.

Example 1 continued

Multiply the entire equation by the LCM of the denominators, then divide out the common factors, which gives:

$$\frac{3}{8} \cdot \frac{8(3x-1)(x+1)}{1} + \frac{1}{3x-1} \cdot \frac{8(3x-1)(x+1)}{1} = \frac{2}{x+1} \cdot \frac{8(3x-1)(x+1)}{1}$$

Multiplying by the LCM of the denominators means that all the denominators in the equation cancel, leaving:

$$3(3x-1)(x+1) + 8(x+1) = 2 \cdot 8(3x-1)$$

$$(9x^2 + 6x - 3) + (8x + 8) = 48x - 16$$

$$9x^2 + 14x + 5 = 48x - 16$$

$$9x^2 - 34x + 21 = 0$$

$$(9x-7)(x-3) = 0$$

$$x = \frac{7}{9} \quad \text{or} \quad x = 3$$

Check it out:

This is a quadratic equation, so you could use the quadratic formula (see Topic 7.3.1) to work out the solutions instead of factoring.

Check it out:

Make sure that these solutions are allowed — that $x \neq \frac{1}{3}$ and $x \neq -1$.



Guided Practice

Solve each of the following fractional equations.

6. $\frac{x-1}{3} - \frac{x+1}{2} = \frac{5}{6}$

7. $3 - \frac{10}{x} = \frac{-3}{x^2}$

8. $a-1 = \frac{5}{a+3}$

9. $5x+1 = \frac{11}{x-1}$

10. $\frac{y}{y-1} = \frac{2}{2y-3}$

11. $\frac{2}{k+3} - \frac{k-6}{k^2-9} = 0$

Always Check that the Solutions Work

It's a **good idea** to check the solutions you have worked out to **make sure** that they are **correct**. The best way to check a solution is to **put it back** into the original equation and **verify** that both sides of the equation are **equal**.

In the problem on the last page, the solutions for the fractional equation

$$\frac{3}{8} + \frac{1}{3x-1} = \frac{2}{x+1} \text{ were found to be } x = \frac{7}{9} \text{ or } x = 3.$$

Check these solutions by putting them back into the equation, **one at a time**.

Check it out:

Remember that dividing a number by a fraction is the same as multiplying that number by the reciprocal of the fraction, for example:

$$a \div \frac{b}{c} = a \times \frac{c}{b} = \frac{ac}{b}$$

Example 1 continued

$$\text{First, check } x = \frac{7}{9}: \quad \frac{3}{8} + \frac{1}{3 \cdot \frac{7}{9} - 1} = \frac{2}{\frac{7}{9} + 1}$$

$$\frac{3}{8} + \frac{1}{\frac{7}{3} - 1} = \frac{2}{\frac{16}{9}}$$

$$\frac{3}{8} + \frac{3}{4} = 2 \cdot \frac{9}{16}$$

$$\frac{9}{8} = \frac{9}{8}$$

Both sides are equal, which means that the solution $x = \frac{7}{9}$ is valid.

$$\text{Now, check } x = 3: \quad \frac{3}{8} + \frac{1}{3 \cdot 3 - 1} = \frac{2}{3 + 1}$$

$$\frac{3}{8} + \frac{1}{8} = \frac{2}{4}$$

$$\frac{1}{2} = \frac{1}{2}$$

Both sides are equal, which means that the solution $x = 3$ is correct.

If you put the **solution** into the equation and find that the two sides are **not equal**, then your solution is **incorrect**.

That means you'll have to **go back** and check each stage of your work.

Example 2

Solve $\frac{3}{2x+1} - \frac{4}{x-1} = -1$, first stating any values of x for which the equation is undefined.

Solution

First take a look at the equation and figure out the **limitations**.

The equation is **undefined** when either $2x + 1 = 0$ or $x - 1 = 0$.

This happens when $x = -\frac{1}{2}$ or $x = 1$.

So the limitations on the solution are that **x cannot equal $-\frac{1}{2}$ or 1**.

Step 1: Find the **least common multiple** of the **denominators**:

$2x + 1$ and $x - 1$.

The LCM for these denominators is $(2x + 1)(x - 1)$.

Step 2: Now, **multiply both sides** of the equation by $(2x + 1)(x - 1)$ to **eliminate** the denominators from the **rational expressions**.

$$\frac{3}{\cancel{2x+1}} \cdot \frac{(\cancel{2x+1})(x-1)}{1} - \frac{4}{\cancel{x-1}} \cdot \frac{(2x+1)(\cancel{x-1})}{1} = -1 \cdot (2x+1)(x-1)$$

Check it out:

The whole equation can be divided by the common factor 2.

Check it out:

Make sure both solutions are allowed, that is $x \neq -\frac{1}{2}$, $x \neq 1$.

Check it out:

Both of these solutions are correct because the equation balances when the solutions are substituted for x .

Example 2 continued

Step 3: Reduce the equation to its **lowest terms**:

$$3 \cdot (x - 1) - 4 \cdot (2x + 1) = -1 \cdot (2x + 1)(x - 1)$$

$$3x - 3 - 8x - 4 = -1 \cdot (2x^2 - x - 1)$$

$$-5x - 7 = -2x^2 + x + 1$$

$$2x^2 - 6x - 8 = 0$$

$$x^2 - 3x - 4 = 0$$

Step 4: Then **factor** to find the **solutions**:

$$(x - 4)(x + 1) = 0$$

$$\text{So, } x - 4 = 0 \text{ or } x + 1 = 0$$

$$\text{So the solutions are: } x = 4 \text{ or } x = -1$$

Now, **check** your solutions to **make sure** that they work.

Put $x = 4$ into the equation:

$$\frac{3}{2 \cdot 4 + 1} - \frac{4}{4 - 1} = -1$$

$$\frac{3}{9} - \frac{4}{3} = -1$$

$$\frac{1}{3} - \frac{4}{3} = -1$$

$$-1 = -1$$

Put $x = -1$ into the equation:

$$\frac{3}{2 \cdot (-1) + 1} - \frac{4}{(-1) - 1} = -1$$

$$\frac{3}{-1} - \frac{4}{-2} = -1$$

$$-3 + 2 = -1$$

$$-1 = -1$$

Guided Practice

Solve each of the following fractional equations.

$$12. \frac{3x}{x+4} - \frac{2x}{x-2} = \frac{x-36}{x^2+2x-8}$$

$$13. \frac{3}{c-4} = \frac{c+2}{c^2-3c-4} + \frac{1}{2(c+1)}$$

$$14. \frac{4}{15} + \frac{2}{x+3} = \frac{3}{2x-1}$$

$$15. \frac{2}{x+4} - \frac{1}{x+3} = 1 - \frac{2}{x^2+7x+12}$$

$$16. \frac{x+1}{2x-3} - 2 = -\frac{x+3}{2x+3}$$

✓ Independent Practice

Solve each of the following fractional equations.

1. $\frac{2}{y+1} = \frac{5}{y-1} - \frac{1}{y+2}$

2. $6 + \frac{1}{x} = \frac{12}{x^2}$

3. $\frac{3}{y} - \frac{y}{y-2} = -2$

4. $\frac{1}{3-x} = -\frac{4x}{x^2-9}$

5. $\frac{2}{2y-1} - \frac{6}{3y-1} = -\frac{1}{y-1}$

6. Solve for y the following equation: $-\frac{3y+8}{y+1} = 2 - \frac{10y+1}{2y-1}$

7. Use your solution to Exercise 6 to find the value of $y^2 - 1$.

8. Find the value of $5k - 7$ if $\frac{1}{k-1} + \frac{2}{2k-1} - \frac{6}{3k-1} = 0$.

9. One number is 8 less than another. The quotient formed by dividing the higher number by the lower number is $\frac{15}{11}$.

Find the values of the two numbers.

10. One integer is 5 less than another.

Find the numbers if the sum of their reciprocals is $\frac{17}{66}$.

11. The denominator of a fraction is 3 more than its numerator.

If the sum of the fraction and its reciprocal is $\frac{29}{10}$, find the fraction.

12. The denominator of a fraction is 2 more than the numerator.

A second fraction is created by subtracting 2 from the numerator and subtracting 3 from the denominator of the first fraction.

Find the two fractions if the first fraction subtracted from the second fraction is $\frac{5}{88}$.

13. Solve the following system of equations by first rearranging them so that they are not in fractional form:

$$\frac{3}{x+2} - \frac{2}{x+3} = \frac{7y}{3x^2 + 15x + 18}$$

$$\frac{4}{y+3} + \frac{1}{y+1} = \frac{11x}{y^2 + 4y + 3}$$

Round Up

The main thing to learn here is that fractional equations can be solved by first **multiplying** the entire equation by the **least common multiple** of the denominators, then **canceling** all common factors.

The equation you're left with can then be **solved** using methods from earlier Chapters.

Remember to **check** that all your solutions actually work by putting them into the original equation.

Applications of Fractional Equations

California Standards:

13.0: Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.

What it means for you:

You'll model real-life situations and solve equations involving rational numbers.

Key words:

- common denominator
- rational
- common factor
- limitation
- undefined
- least common multiple

Don't forget:

Remember: $\text{Time} = \frac{\text{Distance}}{\text{Speed}}$

Check it out:

You need to convert the time from minutes to hours because in the problem, speed is measured in miles per hour.

Now it's time to solve some more fractional equations. This time, the problems are about real-life situations, so you'll have to model them as equations before you can start with the math working.

You Might Be Given a Word Problem to Solve

Here's one long example showing a model of a real-life problem.

Example 1

In a bike race, Natasha cycles along flat ground for 4 miles against a 2 mile per hour wind. After four miles, she turns around and follows the same route back to the start, this time with the wind behind her.

If the journey takes 50 minutes, find how fast Natasha would travel if there were no wind. (Assume that she would travel at a constant speed without a wind.)

Solution

Step 1 — Write the equation.

The **time** Natasha takes to complete the race, can be written as:

$$\text{Time}_{\text{there}} + \text{Time}_{\text{back}} = 50 \text{ minutes}$$

Using this, an equation for Natasha's race time can be written in terms of **distance** and **speed**.

$$\frac{\text{Distance}_{\text{there}}}{\text{Speed}_{\text{there}}} + \frac{\text{Distance}_{\text{back}}}{\text{Speed}_{\text{back}}} = \text{Time}_{\text{total trip}}$$

$$\frac{4}{s-2} + \frac{4}{s+2} = \frac{50}{60} = \frac{5}{6}$$

Where: s = Natasha's speed in no wind
 $s - 2$ = Natasha's speed against the wind
 $s + 2$ = Natasha's speed with the wind behind her
 $s \neq \pm 2$ (for the rational expressions to be defined)

Example 1 continued

Step 2 — Multiply by the LCM.

The **LCM** of the denominators is $6(s - 2)(s + 2)$.

Multiply the equation by the LCM of the denominators to give:

$$24(s + 2) + 24(s - 2) = 5(s - 2)(s + 2)$$

$$24s + 48 + 24s - 48 = 5s^2 - 20$$

$$5s^2 - 48s - 20 = 0$$

Step 3 — Factor and Solve for x .

Now, factor the equation and **solve for x** :

$$(5s + 2)(s - 10) = 0$$

$$\text{So, } s = -\frac{2}{5} \text{ or } s = 10$$

So Natasha's speed if there were no wind would be **10 miles per hour**.

Although $-\frac{2}{5}$ is a **valid** solution of the **algebraic** problem, it isn't a correct answer for this example because speed can only have a **positive value**.

Don't forget:

Remember to check that the solutions work when put back into the original equation.

Guided Practice

1. On Monday, a distribution company shipped a load of oranges in crates, with a total weight of 124 lb. On Tuesday it shipped another load of oranges, also with a total weight of 124 lb. However, on Tuesday there was one crate fewer than on Monday, so each crate was $\frac{1}{8}$ lb heavier. How many crates were shipped on Monday?
2. Rose spent \$2.40 on pens. If each pen had cost 4 cents more, she would have been able to buy 10 fewer pens for the same money. How many pens did Rose buy?
3. Dajanique bought x boxes of candy for a total of \$1.26. She kept four boxes and sold the rest for a total of \$1.40. If she sold each box for 3 cents more than it cost her, how many boxes did she buy?
4. A teacher spent \$8.40 on sets of chapter tests. If each set of tests had been 2 cents less, the teacher would have gotten two extra sets for the same price. How many sets did the teacher get?

Independent Practice

1. Vanessa bought a set of military medals through an antique dealer for \$120. She gave two of the medals to her father and resold the rest, charging \$4 more per medal than she paid. If Vanessa sold the medals for a total of \$156, how much did she charge for each medal?
2. Hearst Castle is 180 miles from Ventura by road. Two coaches leave Ventura at the same time, both heading for Hearst Castle, but one averages 5 mph faster than the other. If the faster coach reaches Hearst Castle half an hour earlier than the slower coach, what is the average speed of the faster coach?
3. A cyclist completes a 150-mile race in a certain amount of time. She completes another 150-mile race a month later, but this time it takes an hour longer to cover the same distance and her average speed is 5 mph less than in the first race. Find the average speed for the cyclist during her first race.
4. Juan's band produced a number of CDs to sell at a gig. The batch of CDs cost \$170.00 to make and each CD was sold for \$2.50 more than it cost to produce. Juan gave 2 CDs to his friends, but sold the rest for a total of \$198.00. How many CDs did Juan produce?
5. Chen bought a bag of groceries weighing 15 pounds. His friend, Jo, bought a bag of groceries that also weighed 15 pounds, but contained one less item. The average weight per item for Jo's groceries was $\frac{1}{2}$ pound more than for Chen's. How many items were in Jo's grocery bag?
6. José ordered a box of fruits for his market stand costing \$6. When his order arrived, José discovered that 20 fruits were rotten and threw them in the trash. He sold all of the remaining fruits for a total of \$8, charging 3 cents more for each fruit than he paid for it. How many fruits did José order to begin with?
7. A wholesaler bought a batch of T-shirts for \$77.00. She gave two of the T-shirts to her daughters and then sold the rest for a total of \$90. If the wholesaler sold each T-shirt for \$2 more than it cost her, how much did she pay for each T-shirt?

Round Up

The only difference between solving word problems and answering straightforward algebra questions is that you have to write the equations yourself for a word problem. After that, they are solved in exactly the same way.

Topic 8.5.1

Section 8.5 Relations

California Standards:

16.0: Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.
17.0: Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.

What it means for you:

You'll find out what relations are, and some different ways of showing relations.

Key words:

- relation
- ordered pair
- domain
- range

Check it out:

In an ordered pair (x, y) , the first variable, x , is usually called the **independent variable**, and the second, y , is the **dependent variable** — it depends on the first. For example, in the relation $y = 4x^2 - 3x$, the values of y in the range depend on the values of x in the domain.

Check it out:

The **order** you write the elements of the domain or range isn't important, so you could write the range from Example 1 as $\{2, 4, 5, 7, 8\}$. Also, if an element is connected to more than one element in the other set, you only need to write it down once.

Relations in Math are nothing to do with family members.

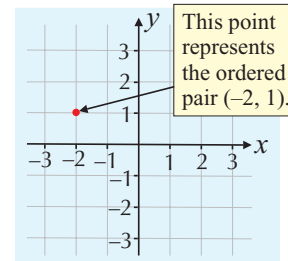
They're useful for describing how the x and y values of coordinate pairs are linked.

A Relation is a Set of Ordered Pairs

Before you can define a relation, you need to understand what “**ordered pairs**” are:

An ordered pair is just two numbers or letters, (or anything else) written in the form (x, y) .

If x and y are both real numbers, ordered pairs can be plotted as **points** on a **coordinate plane**, where the first number in the ordered pair represents the x -coordinate and the second number represents the y -coordinate.



A relation is any **set** of **ordered pairs**. Relations are represented using **set notation**, and can be named using a letter: for example: $m = \{(1, 4), (2, 8), (3, 12), (4, 16)\}$.

Every relation has a **domain** and a **range**.

Domain: the set of all the first elements (x -values) of each ordered pair, for example: domain of $m = \{1, 2, 3, 4\}$

Range: the set of all the second elements (y -values) of each ordered pair, for example: range of $m = \{4, 8, 12, 16\}$

An important point to note is that there may or may not be a **reason** for the pairing of the x and y values. Looking at the relation m , above, you can see that the x and y values are related by the equation $y = 4x$ — but **not all** relations can be described by an equation.

Example 1

State the domain and range of the relation

$$r = \{(1, 4), (3, 7), (3, 5), (5, 8), (9, 2)\}$$

Solution

$$\text{Domain} = \{1, 3, 5, 9\}$$

$$\text{Range} = \{4, 7, 5, 8, 2\}$$

Example 2

State the domain and range of the relation

$$f = \{(a, 2), (b, 3), (c, 4), (d, 5)\}.$$

Solution

Domain = $\{a, b, c, d\}$ Range = $\{2, 3, 4, 5\}$

Ordered pairs are not the only way to represent relations.

✓ Guided Practice

State the domain and range of each relation.

1. $f(x) = \{(1, 1), (-2, 1), (3, 5), (-3, 10), (-7, 12)\}$

2. $f(x) = \{(-1, -1), (2, 2), (3, -3), (-4, 4)\}$

3. $f(x) = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$

4. $f(x) = \{(a, b), (c, d), (e, f), (g, h)\}$

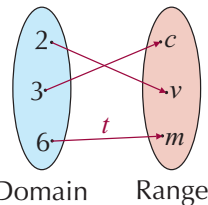
5. $f(x) = \{(-1, 0), (-b, d), (e, 3), (7, -f)\}$

6. $f(x) = \{(a, -a), (b, -b), (-c, c), (\frac{1}{2}, -j)\}$

Mapping Diagrams Can Be Used to Represent Relations

One way to visualize a relation is to use a **mapping diagram**. In the diagram, the area on the left represents the domain, while the area on the right represents the range.

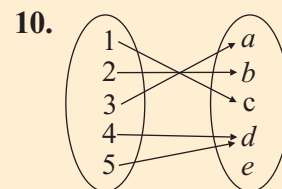
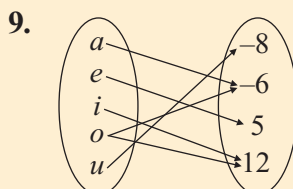
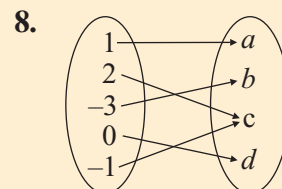
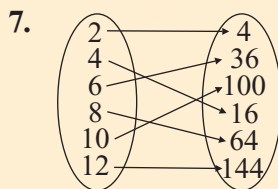
The **arrows** show which member of the domain is paired with which member of the range.



This mapping diagram represents the relation $t = \{(2, v), (3, c), (6, m)\}$.

✓ Guided Practice

State the domain and range of each relation.



You Can Use Input-Output Tables

In input-output tables, the input is the **domain** and the output is the **range**.

This table represents the relation $\{(1, 1), (2, 3), (3, 6), (4, 10), (5, 15)\}$.

Input	Output
1	1
2	3
3	6
4	10
5	15

Guided Practice

State the domain and range of each relation.

11.

Input	Output
1	32
5	6
12	0.3

12.

Input	Output
-2	8
-1	5
2	8
3	13

13.

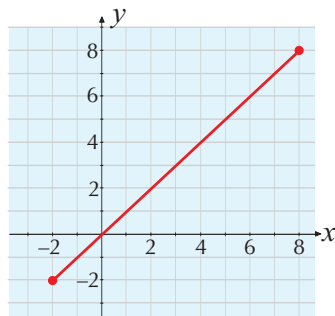
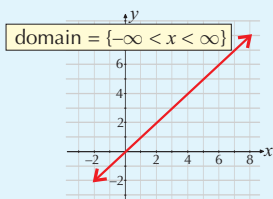
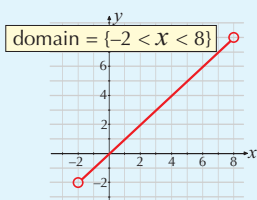
Input	Output
-3	-26
-1	0
0	1
1	2

Relations Can Be Plotted as Graphs

Relations can be plotted on a coordinate plane, where the domain is represented on the x -axis and the range on the y -axis. Graphs are most useful when you have **continuous** sets of values for the **domain** and **range**, so that you can connect points with a smooth curve or straight line.

Check it out:

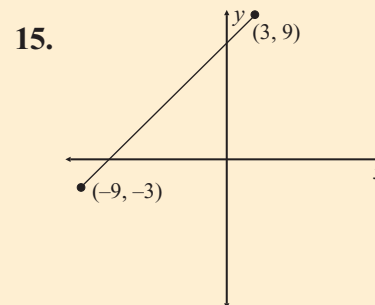
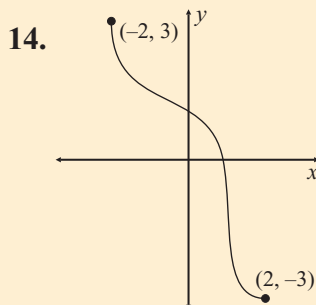
The filled circles mean that the domain includes the values -2 and 8 . A hollow circle would mean that the value was not included, and an arrowhead would mean that the domain continued to infinity in that direction. So, for example:



This graph represents the relation $\{(x, y = x)\}$ with domain $\{-2 \leq x \leq 8\}$.

Guided Practice

State the domain and range of each relation.



✓ Independent Practice

Define each of the following terms.

1. Relation

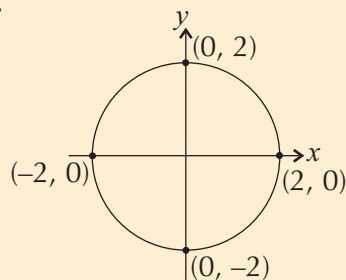
2. Range

3. Domain

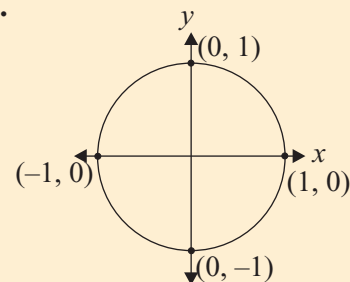
In Exercises 4–15, state the domain and range of each relation.

4. $\{(x, x^2) : x \in \{-1, 2, 4\}\}$

5.



6.



7. $f(x) = \{(x, \sqrt{2x-5}) : x \in \{3, 4.5, 7\}\}$

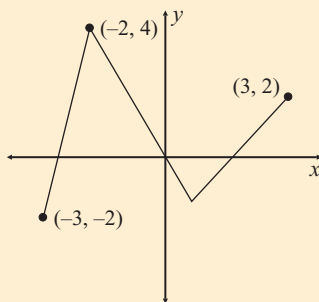
8. $f(x) = \{(x, x^2 - 1) : x \in \{0, 1, 2\}\}$

9. $f(x) = \{(x, -x^2 + 3) : x \in \{-2, -1, 0, 1, 2\}\}$

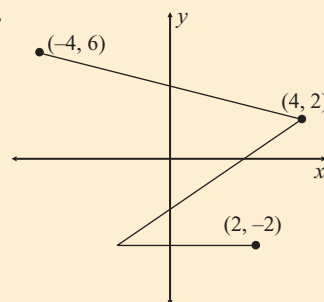
10. $f(x) = \{(x, \frac{x}{x-1}) : x \in \{-1, 0, 2\}\}$

11. $f(x) = \{(x, \frac{x+1}{x+2}) : x \in \{-1, 0, 2, 3\}\}$

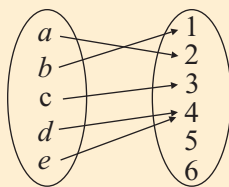
12.



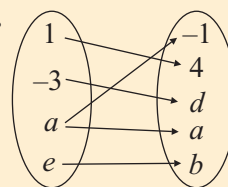
13.



14.



15.



Round Up

The important thing to remember is that a relation is just a set of ordered pairs showing how a domain set and a range set are linked.

Functions

California Standards:

16.0: Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.

18.0: Students determine whether a relation defined by a graph, a set of ordered pairs, or a symbolic expression is a function and justify the conclusion.

What it means for you:

You'll find out what functions are, and you'll say whether particular relations are functions.

Key words:

- function
- relation
- ordered pair
- domain
- range

A function is a special type of relation.

Functions Map from the Domain to the Range

A **relation** is **any** set of ordered pairs — without restriction.

A **function** is a type of relation that has the following restriction on it:

A **function** is a set of ordered pairs (x, y) such that no two ordered pairs in the set have the same x -value but different y -values. That is, each member of the domain maps to a **unique** member of the range.

Example 1

Determine whether each of the following relations is a function or not. Justify your answers.

- a) $k = \{(0, 0), (1, 1), (2, 4), (3, 9)\}$
- b) $m = \{(1, 2), (2, 5), (1, 4), (3, 6)\}$
- c) $p = \{(-1, -3), (0, -1), (1, 1), (2, 3)\}$
- d) $v = \{(-2, 5), (-1, 5), (0, 5), (1, 5), (5, 5)\}$

Solution

- a) **k is a function.**
No two different ordered pairs have the same x -value.
- b) **m is not a function.**
The ordered pairs $(1, 2)$ and $(1, 4)$ have the same first entry, but different second entries.
- c) **p is a function.**
No two different ordered pairs have the same x -value.
- d) **v is a function.**
No two different ordered pairs have the same x -value.

✓ Guided Practice

State whether each relation in Exercises 1–4 is a function or not. Explain your reasoning.

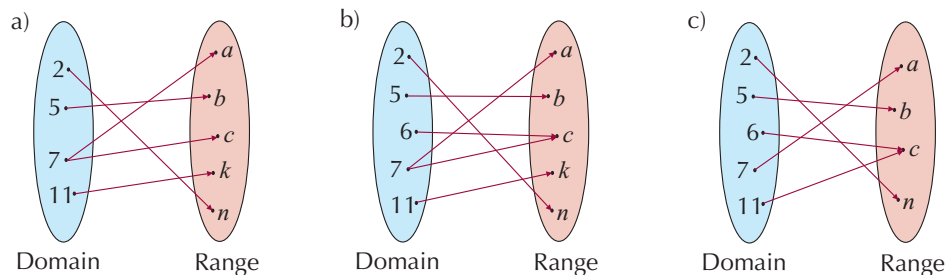
- 1. $m = \{(1, 1), (2, 8), (3, 27)\}$
- 2. $b = \{(a, 1), (b, 2), (c, 3), (a, 4)\}$
- 3. $v = \{(7, 1), (7, 2), (7, 7)\}$
- 4. $t = \{(1, 7), (2, 7), (7, 7)\}$

Functions Can Be Represented in Different Ways

Relations do not have to be written as lists of ordered pairs for you to be able to identify functions.

Example 2

Determine whether each of the mappings below represents a function.



Solution

a) and b) are **NOT** functions, since 7 is mapped to two different values — $(7, a)$ and $(7, c)$ have the same x -value.

c) **IS** a function. Each member of the domain only maps onto one member of the range.

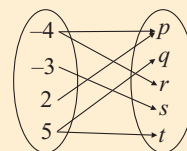
✓ Guided Practice

The mapping on the right shows the relation $g(x)$.

- State the domain and range of the relation.
- Is the relation a function? Explain your answer.
- Find $g(0)$.

The mapping on the right shows the relation $h(x)$.

- State the domain and range of the relation.
- Is this relation a function? Explain your answer.
- Find $h(-3)$.



Don't forget:

Remember that this isn't always the case — some functions cannot be expressed as an equation.

Functions are Often Written as Equations

Some functions can be expressed as an **equation**.

For this to be possible, there must be a **reason** for the pairing between each member of the domain and each member of the range.

The **equation** represents the way the members of the domain and range are **paired**.

Example 3

Express the following function as an equation:

$$f = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$$

Solution

The relationship between the x -values and y -values is $y = x^2$.

The domain of the function is $x \in \{1, 2, 3, 4\}$.

The function can be written:

$$f = \{(x, y) \text{ such that } y = x^2 \text{ and } x \in \{1, 2, 3, 4\}\}$$

Check it out:

The symbol \in is shorthand for “is a member of the set.” It just means that x takes only those values.

You will often see functions written in the form $y = x^2$, without any domain specified. By convention, you then take the domain to be **all** values of x for which the function is **defined**.

✓ Guided Practice

Express the following functions in terms of an equation.

11. $f = \{(-4, 0), (-3, 1), (0, 4), (1, 5), (2, 6)\}$

12. $g = \{(-2, 5), (0, 1), (1, 2), (2, 5)\}$

13. $h = \{(-5, -4.5), (-3, -2.5), (1, 1.5), (3, 3.5), (5, 5.5)\}$

14. $f = \{(-3, -27), (-2, -8), (-1, -1), (0, 0), (1, 1), (2, 8)\}$

15. $g = \{(-2, 8), (0, 0), (1, 2), (2, 8), (3, 18)\}$

16. $h = \{(-3, 17), (-1, 1), (0, -1), (1, 1), (2, 7)\}$

The Vertical Line Test Shows if a Graph is a Function

By definition, a function cannot have any two ordered pairs that have the same first coordinate but different second coordinates, i.e. for each value of x there is only one possible value of y .

Graphically, this means that no vertical line can intersect the graph of a function at **more than one point**.

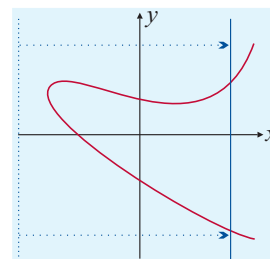
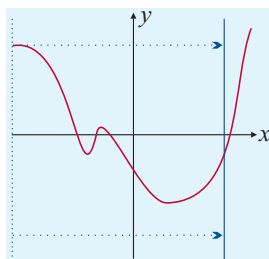
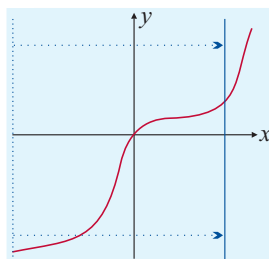
Vertical Line Test to determine whether a graph represents a function:

Simply hold a **straightedge** parallel to the y -axis at the far left-hand side of the graph, then move it horizontally along the graph from left to right.

If, at any position along the x -axis, it is possible for you to draw a vertical line that **intersects** with the graph **more than once**, then the graph does not represent a function.

Example 4

Use the vertical line test to determine whether the following graphs represent functions.

**Solution**

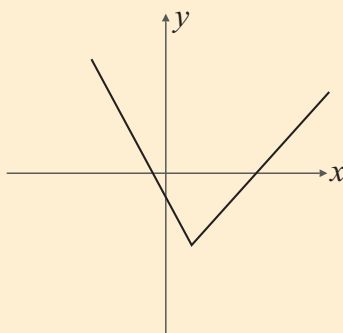
The **first two graphs** pass the vertical line test — you cannot draw a vertical line that intersects with either graph at more than one point, so they **represent functions**.

The **third graph** does **not** represent a function — the vertical line test fails.

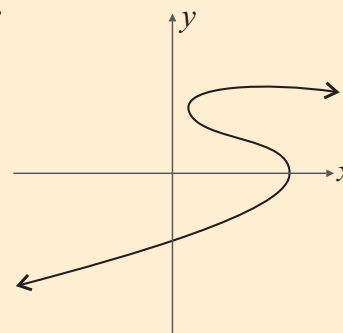
✓ Guided Practice

In Exercises 17–20, use the vertical line test to determine whether each graph represents a function or not.

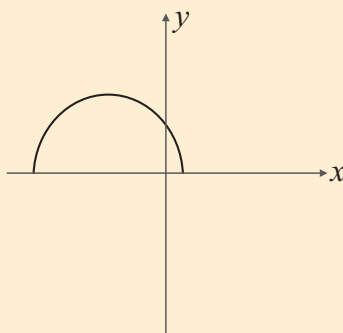
17.



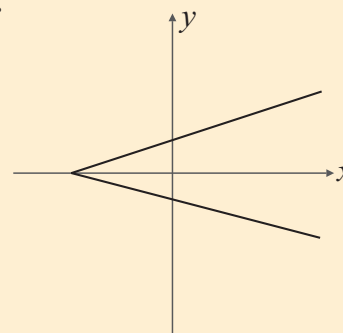
18.



19.



20.



✓ Independent Practice

1. Define a function and give an example.

In Exercises 2–7, use the given relation with the domain

$x = \{-2, -1, 0, 1, 2\}$ to generate sets of ordered pairs.

Use them to determine whether the relation is a function or not.

2. $m = \{(x, x^2 - 4)\}$

3. $t = \{(x, x + 2)\}$

4. $k = \{(x, y = (x - 2)(x + 2))\}$

5. $p = \{(x, y = \pm\sqrt{4 - x^2})\}$

6. $j = \{(x, y = 2x - 1)\}$

7. $b = \{(x, y = x \pm (3x - 4))\}$

8. In the equation $x^2 + y^2 = 9$, is y a **function** of x ?

Explain your reasoning.

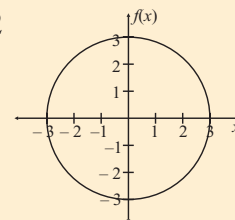
Using the graph on the right, answer Exercises 9–12 about relation f .

9. State the domain and range of the relation.

10. Is the relation a function?

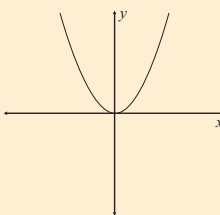
11. Find the value(s) of $f(0)$.

12. Find the value(s) of $f(3)$.

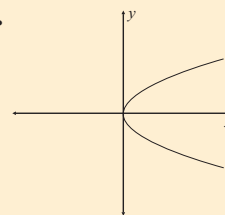


Use the vertical line test to determine if the graphs below are functions.

13.



14.



For Exercises 15–16, find the range (y) for each relation when the domain is $\{-4, -2, 0, 2, 4\}$, and determine whether the relation is a function.

15. $y = \frac{1}{2}x + 1$

16. $y = x - 6$

For Exercises 17–19, find the domain (x) for each relation when the range is $\{-6, -3, 0, 3, 6\}$, and determine whether the relation is a function.

17. $y = \frac{1}{3}x - 2$

18. $y = x + 5$

19. $y = \pm\sqrt{x^2}$

20. Are all quadratics of the form $y = ax^2 + bx + c$ and $y = -ax^2 + bx + c$ functions? Explain your answer.

21. Are circles functions? Explain your answer.

Round Up

*Functions are special types of relations. That means that **all functions are relations** — but **not all relations are functions**. A relation is only a function if it maps **each** member of the domain to **only one** member of the range.*

Function Notation

Functions are often written using *function notation*.

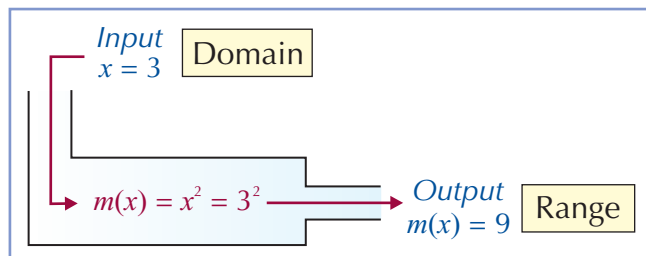
Think of a Function as a “Rule” or “Machine”

The **function** m is a rule that assigns to each value x of its domain a distinct value $m(x)$ of its range.

Think of m as a **machine** that processes x according to some rule and outputs the value $m(x)$.

Suppose $m(x) = x^2$. You could draw the following diagram to represent how the function processes any given value of x .

Function machine: $m(x) = x^2$



$m(x) = x^2$ takes a value of x from the domain, in this case $x = 3$, squares it, $3^2 = 9$, and outputs a value of the range, 9.

When you are referring to a **particular value** of a function, you replace the x with the relevant number — so in the example above, the value of m when $x = 3$ would be denoted $m(3)$.

Similarly, you can substitute expressions into a function, so for example you could find $m(x+3) = (x+3)^2$ by replacing x with $(x+3)$ in the function.

California Standards:

16.0: Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.

17.0: Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.

What it means for you:

You'll see some different ways of representing functions.

Key words:

- function
- domain
- range

Check it out:

The notation $m(x)$ is read as “ m of x ” or “the value of function m at x .”

Example 1

Given the function $f(x) = 2x^2 - 1$, find

a) $f(-1)$, b) $f(3)$, and c) $f(x + a)$.

Solution

$$\text{a) } f(-1) = 2 \cdot (-1)^2 - 1 = 2 \cdot 1 - 1 = 2 - 1 = 1 \quad \therefore f(-1) = 1$$

$$\text{b) } f(3) = 2 \cdot 3^2 - 1 = 2 \cdot 9 - 1 = 18 - 1 = 17 \quad \therefore f(3) = 17$$

$$\text{c) } f(x + a) = 2 \cdot (x + a)^2 - 1 = 2(x^2 + 2ax + a^2) - 1 = 2x^2 + 4ax + 2a^2 - 1$$

Example 2

If $P(x) = x^3 + 1$, find the range of $P(x)$ when the domain of $P(x)$ is the set $\{-2, -1, 0, 1, 2\}$.

Solution

The range is the set of all values of $P(x)$ for which $x \in \{-2, -1, 0, 1, 2\}$.

$$\begin{aligned} \text{So, range} &= \{P(-2), P(-1), P(0), P(1), P(2)\} \\ &= \{(-2)^3 + 1, (-1)^3 + 1, (0)^3 + 1, (1)^3 + 1, (2)^3 + 1\} \\ &= \{-7, 0, 1, 2, 9\} \end{aligned}$$

✓ Guided Practice

For Exercises 1–6, let $f(x) = 3x - 1$ and $g(x) = x^2 - 2$.

Find each of the values indicated.

1. $f(-2)$

2. $f\left(\frac{1}{3}\right)$

3. $g(a)$

4. $f(a - 1)$

5. $f(3) \cdot g(-2)$

6. $g(k) - f(k)$

✓ Independent Practice

1. The function $f(t) = \frac{5}{9}(t - 32)$ is used to convert temperatures from degrees Fahrenheit to degrees Celsius. Find $f(212)$.

2. Supposing $f(x) = -2x^2 + x - 5$, evaluate $f(x)$ for $x \in \{-1, 2, b\}$.

In Exercises 3–5, find $f(x + h)$ for each of the given functions.

3. $f(x) = 2x - 1$

4. $f(x) = x^2 - x - 2$

5. $f(x) = -3x + h$

Use the functions $f(x) = 2x + 1$ and $g(x) = x^2 + 2x + 1$ to find the value of each expression in Exercises 6–9.

6. $f(-1) + g(2)$

7. $g(-1) - f(3)$

8. $f(x + h) + g(x + h)$

9. $f(x + h) - g(x + h)$

Round Up

There's nothing too complicated here — the main thing is to recognize that if you see the notation $f(\text{something})$, then it's likely to be a function.

More on Functions

California Standards:

16.0: Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.

17.0: Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.

What it means for you:

You'll show whether two functions are equal.

Key words:

- function
- ordered pair
- domain
- range

This Topic's all about the special rules for telling whether two functions are equal.

Equality of Functions

A function f is **equal** to another function g **if, and only if**, the set of ordered pairs of f is **identical** to the set of ordered pairs of g .

It isn't enough for the two functions to be represented by equivalent equations [for example, $y = 2x + 4$ and $y = 2(x + 2)$].

For the two sets of ordered pairs to be identical, the two functions must also have the **same domain** (and therefore the same range).

Example 1

- a) Determine whether the following two functions are equal:

$$m = \{(x, y), \text{ such that } y = x^2 - 4 \text{ and } x \in \{-1, 0, 2, 3\}\}$$

$$b = \{(x, y), \text{ such that } y = (x - 2)(x + 2) \text{ and } x \in \{-1, 0, 2, 3\}\}$$

- b) State the range of m .

Solution

- a) Find each set of ordered pairs by substituting each value of x into the equation for y .

$$\begin{aligned} m &= \{(-1, [-1]^2 - 4), (0, 0^2 - 4), (2, 2^2 - 4), (3, 3^2 - 4)\} \\ &= \{(-1, -3), (0, -4), (2, 0), (3, 5)\} \end{aligned}$$

$$\begin{aligned} b &= \{(-1, [-1 - 2][-1 + 2]), (0, [0 - 2][0 + 2]), \\ &\quad (2, [2 - 2][2 + 2]), (3, [3 - 2][3 + 2])\} \\ &= \{(-1, -3), (0, -4), (2, 0), (3, 5)\} \end{aligned}$$

Since each ordered pair for m is in b and vice versa, then **the functions m and b are equal**.

- b) The range of $m = \text{range of } b = \{-3, -4, 0, 5\}$

Example 2

The domain of both of the following functions is the set of all real numbers.

$$p = \{(x, y = x^3 + 3x^2 + 3x + 1)\}$$

$$q = \{(x, y = (x + 1)^3)\}$$

Determine whether the two functions are equal.

Solution

Both functions have the same domain, so if the equations that generate the y -values are equal, then the functions must be equal.

$$\begin{aligned} \text{Expand } (x + 1)^3: \quad (x + 1)^3 &= (x + 1)(x + 1)(x + 1) \\ &= (x^2 + 2x + 1)(x + 1) \\ &= x^3 + 3x^2 + 3x + 1 \end{aligned}$$

So, p and q are equal.

Guided Practice

Determine whether the pairs of relations/functions below are equal.

1. $A = \{(x, y), \text{ such that } y = 2x + 8 \text{ and } x \in \{-1, 0, 1, 2\}\}$

$B = \{(x, y), \text{ such that } 2y = 4x + 16 \text{ and } x \in \{-1, 0, 1, 2\}\}$

2. $A = \{(x, y), \text{ such that } y = x^2 \text{ and } x \in \{1, 4, 9, 16\}\}$

$B = \{(x, y), \text{ such that } y^2 = x \text{ and } x \in \{1, 4, 9, 16\}\}$

Determine whether the pairs of relations/functions below are equal.

The domain of each relation/function is the set of real numbers.

3. $M = \{(x, y = (x + 4)^2)\}$ $N = \{(x, y = x^2 + 4x + 4)\}$

4. $M = \{(x, y = x^2 + 1)\}$ $N = \{(x, y = (x + 1)^2)\}$

5. $M = \{(x, y = x^4(x + 1)(2x - 3))\}$ $N = \{(x, y = 2x^6 - x^5 - 3x^4)\}$

6. $M = \{(x, y = x(3x - 2)(3x + 1))\}$ $N = \{(x, y = 9x^3 + 3x^2 - 2x)\}$

More Function Examples**Example 3**

Suppose $v(x) = \frac{x^2 + 1}{x - 1}$.

a) State any restrictions on x . b) Find $v(-3)$.

Solution

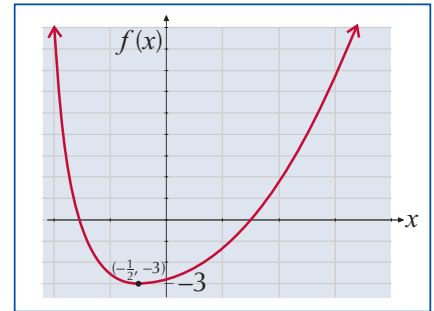
a) The function is undefined when its denominator $(x - 1)$ is 0, so $x \neq 1$.

b) Substitute $x = -3$: $v(-3) = \frac{(-3)^2 + 1}{-3 - 1} = \frac{9 + 1}{-4} = -\frac{10}{4} = -\frac{5}{2}$

So, $v(-3) = -\frac{5}{2}$

Example 4

Determine the range of the function represented by the graph on the right. Explain your answer.

**Solution**

The lowest value of $f(x)$ is -3 .

The values of $f(x)$ get infinitely large as x gets larger (in both the positive and negative x directions) — as indicated by the arrowheads on the graph.

$$\text{Range} = \{f(x) : f(x) \geq -3\}$$

Check it out:

A colon means “such that,” so you would read this solution as “the set of all $f(x)$ such that $f(x)$ is greater than or equal to -3 .”

Example 5

Determine the domain of $f(x) = \sqrt{2x - 6}$, given that it contains all real x for which $f(x)$ is defined. Explain your thinking.

Solution

Any expression \sqrt{n} is defined for all real values of $n \geq 0$.

$$\begin{aligned} \text{So, for the function to be defined, } 2x - 6 &\geq 0 \\ &\Rightarrow 2x \geq 6 \\ &\Rightarrow x \geq 3 \end{aligned}$$

So, the **domain of the function** = $\{x : x \in \mathbb{R} \text{ and } x \geq 3\}$

Don't forget:

\mathbb{R} is the set of all real numbers.

Example 6

Given that $g(x) = x^2 - 1$, find $\frac{g(x+h) - g(x)}{h}$ in terms of x and h .

Solution

$$\begin{aligned} &= \frac{[(x+h)^2 - 1] - [x^2 - 1]}{h} \\ &= \frac{x^2 + 2xh + h^2 - 1 - x^2 + 1}{h} \\ &= \frac{2xh + h^2}{h} \\ &= \frac{h \cdot (2x + h)}{h} \\ &= 2x + h \end{aligned}$$

Substitute in the functions $g(x)$ and $g(x+h)$

Expand and simplify

✓ Guided Practice

In Exercises 7–10, find the values in terms of h (and x , where appropriate).

7. Supposing $f(x) = x^2 - 2$, find $f(h + 2) - f(2)$.

8. If $f(x) = 2x^2 + 4x - 6$, find $f(h - 3) + f(2h)$.

9. Supposing $f(x) = x^2$, find $\frac{f(x+h) - f(x)}{h}$.

10. If $m(x) = 2x - 3$, find the value of $\frac{m(x+h) - m(x)}{h}$.

Determine the domain for the functions in Exercises 11 and 12, given that they contain all real numbers for which the functions are defined.

11. $f(x) = \frac{x^2 + 8x + 16}{x^2 - 4x + 4}$

12. $f(x) = \sqrt{\frac{x^2 + 4}{x^2 - 9}}$

✓ Independent Practice

For the functions in Exercises 1–4, state any restrictions and find $f(-1)$.

1. $f(x) = \frac{1}{x} + 3x + 2$

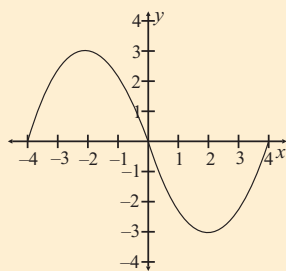
2. $f(x) = \frac{x^2 - 6x - 9}{3x^2 + 11x - 4}$

3. $f(x) = \sqrt{2x + 6}$

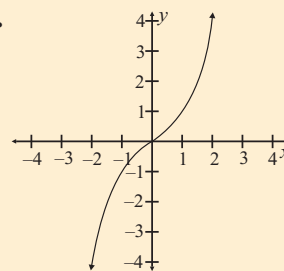
4. $f(x) = \sqrt{x^2 - 1}$

Determine the range of the functions in Exercises 5–6.

5.



6.



7. If $f(x) = x^2 + 2x - 1$, find $f(x + h) - f(x)$.

8. If $g(x) = 2x^2 + x - 4$, find $g(x + h) - g(x)$.

9. If $f(x) = (x + 1)^2 + x$, find $f(2h) - f(2)$.

10. If $f(x) = x^2 + 2x$, find $\frac{f(x+h) - f(x)}{h}$.

11. If $f(x) = x^2 - 2x + 1$, find $\frac{f(x+h) - f(x)}{h}$.

12. Give an example of two equal functions.

Round Up

There are two methods to determine whether two functions are equal. If you're given a small number of values for the domain, you can **substitute** each value of the domain into both functions and see if you get **identical ordered pairs**. The alternative is to show that the two functions can be represented by **equivalent equations** — and if the **domains** are also equal, the **functions** will be equal.

Chapter 8 Investigation

Transforming Functions

In this Investigation you'll practice interpreting information about *functions* using *graphs*.

On a set of axes, draw the graph of the following function:

$$f = \{(x, y) \text{ such that } y = |x| \text{ and } x \in \mathbb{R}\}$$

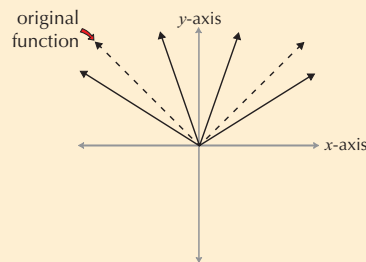
Investigate the functions: $f = \{(x, y) \text{ such that } y = |x| + a, x \in \mathbb{R}, \text{ and } a \in \mathbb{R}\}$,

$$\text{and } f = \{(x, y) \text{ such that } y = |x + a|, x \in \mathbb{R}, \text{ and } a \in \mathbb{R}\}.$$

Write down a **general rule** for what happens.

Extension

- 1) The graph of the function $f = \{(x, y) \text{ such that } y = |x|\}$ can be **stretched** or **compressed**. The diagram on the right shows a horizontal compression and stretch.



Investigate how the function must be changed to **stretch** or **compress** the graph. Write down a **general rule**.

- 2) Investigate how the function $f = \{(x, y) \text{ such that } y = |x|\}$ must be changed so that the graph is **reflected in the x-axis**.

Open-ended extension

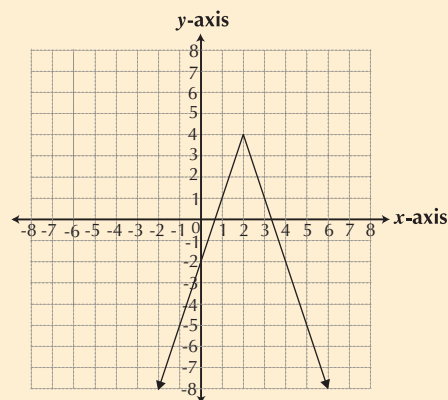
- 1) The graph on the right has been produced by performing a **series of different transformations** on the graph of $f = \{(x, y) \text{ such that } y = |x|\}$.

Identify the **new function** that is represented by the graph.

- 2) Perform a **series of transformations** on the graph of $f = \{(x, y) \text{ such that } y = |x|\}$, recording each one.

Identify the function that your graph represents.

Check that you are correct by creating a **table of x- and y-values** for your function and **plotting the points**.



Round Up

The rules about *transforming the absolute value function* also apply to other *graphs of functions* — so this Investigation is really useful for all sorts of math problems.

Appendixes

Glossary	438
Formula Sheet	440
Index	442

Glossary

Symbols

∞	infinity	\emptyset	the empty set
$<$	is less than	\cap	the <i>intersection</i> of sets
$>$	is greater than	\cup	the <i>union</i> of sets
\therefore	therefore	\subset	is a subset of
\perp	is perpendicular to	$\not\subset$	is not a subset of
\parallel	is parallel to	\in	is an element of
\leq	is less than or equal to	\notin	is not an element of
\geq	is greater than or equal to	\Rightarrow	implies
\neq	is not equal to	\mathbb{R}	the <i>real numbers</i>
\mathbb{I}	the <i>irrational numbers</i>	\mathbb{W}	the <i>whole numbers</i>
\mathbb{N}	the <i>natural numbers</i>	\mathbb{Z}	the <i>integers</i>
\mathbb{Q}	the <i>rational numbers</i>		

A

absolute value the distance between zero and a number on the number line (the absolute value of a is written $|a|$)

algebraic expression a mathematical expression containing at least one *variable*

associative properties (of addition and multiplication)

$$\text{for any } a, b, c: \quad a + (b + c) = (a + b) + c \\ a(bc) = (ab)c$$

B

base in the expression b^x , the base is b

binomial a *polynomial* with two *terms*

C

closed interval an interval that includes its endpoints

closure properties (of real-number addition and multiplication) when two *real numbers* are added or multiplied, the result is also a real number

common factor a number or expression that is a *factor* of two or more other numbers or expressions

commutative properties (of addition and multiplication) for any a, b : $a + b = b + a$ and $ab = ba$

completing the square the process of changing a *quadratic* expression into a *perfect square trinomial*

compound inequality two inequalities combined using either “and” (a *conjunction*) or “or” (a *disjunction*)

D

degree (of a monomial) the *sum* of the powers of the *variables*

degree of a polynomial the largest *degree* of a *polynomial's* terms

denominator the bottom expression of a fraction

dependent system of equations a *system of equations* with infinitely many possible solutions

discriminant for a *quadratic equation* $ax^2 + bx + c = 0$, the discriminant is $b^2 - 4ac$

distributive property (of multiplication over addition)

$$\text{for any } a, b, c: \quad a(b + c) = ab + ac$$

domain (of a relation or function) the set of all possible “inputs” of a *relation* or *function*

E

equivalent equations equations that have the same solution set

equivalent fractions fractions are equivalent if they have the same value

exponent in the expression b^x , the exponent is x

F

factoring writing a *polynomial* as a *product* of two or more *factors*

factor a number or expression that can be multiplied to get another number or expression — for example, 2 is a factor of 6, because $2 \times 3 = 6$

function a rule for transforming an “input” into a unique “output”

G

greatest common factor (GCF) largest expression that is a *common factor* of two or more other expressions; all other common factors will also be *factors* of the GCF

grouping symbols symbols that show the order in which mathematical operations should be carried out — such as parentheses and brackets

I

identities (of addition and multiplication)

the *additive identity* is 0 (zero) — 0 can be added to any other number without changing it;

the *multiplicative identity* is 1 — any number can be multiplied by 1 without changing

inconsistent system of equations a *system of equations* with no solutions

integers the numbers $0, \pm 1, \pm 2, \pm 3, \dots$;
the set of all integers is denoted \mathbb{Z}

intersection (of sets) the intersection of two or more sets is the set of elements that are in all of them;
intersection is denoted by \cap

inverses a number's *additive inverse* is the number that can be added to it to give 0 (the *additive identity*);

a number's *multiplicative inverse* is the number that it can be multiplied by to give 1 (the *multiplicative identity*)

irrational numbers the set of all numbers that cannot be written as a fraction $\frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$; the set of all irrational numbers is denoted \mathbb{I}

L

least common multiple (LCM) the smallest expression that has two or more other expressions as *factors*

like terms two or more *terms* that contain the same *variables*, and where each variable is raised to the same power in every term

linear equation in x and y an equation that can be written in the form $Ax + By = C$ (where A and B are not both zero)

M

monomial an expression with a single term

N

natural numbers the set of numbers 1, 2, 3, ...; the set of all natural numbers is denoted \mathbb{N}

numerator the top expression of a fraction

numeric expression a number or an expression containing only numbers (and therefore no *variables*)

O

open interval an interval that does not contain its endpoints

ordered pair a pair of numbers or expressions written in the form (x, y) ; they can be used to identify a point in the coordinate plane

P

parallel lines lines with equal *slopes*; lines in the same plane that never meet

perfect square trinomial a *trinomial* that can be written as the square of a *binomial*

perpendicular lines lines whose *slopes* multiply together to give the *product* -1 ; lines that intersect at 90°

point-slope formula an equation of a line of the form $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) are the coordinates of a particular point lying on that line

polynomial a *monomial* or *sum* of monomials

prime factorization a factorization of a number where each *factor* is a *prime number*

prime number a number that can only be divided by itself and 1

product the result of multiplying numbers or expressions together

properties of equality

addition property of equality: if $a = b$, then $a + c = b + c$

multiplication property of equality: if $a = b$, then $ac = bc$

subtraction property of equality: if $a = b$, then $a - c = b - c$

division property of equality: if $a = b$, then $a \div c = b \div c$

Q

quadratic equation a *polynomial* equation of *degree* 2

R

radical an expression written with a radical symbol (for example, $\sqrt{2}$)

radicand the number inside a *radical* symbol

range (of a relation or function) the set of all possible “outputs” of a *relation* or *function*

rational expression an expression written as a fraction

rational numbers the set of all numbers that can be written as a fraction $\frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$; the set of all rational numbers is denoted \mathbb{Q}

real numbers denoted \mathbb{R} , all numbers of the number line

reciprocal the multiplicative *inverse* of an expression

relation any set of *ordered pairs* (the first number or expression in each pair can be thought of as the relation’s “input,” the second as the relation’s “output”)

roots of an equation an equation’s roots are its solutions

S

sign of a number whether a number is positive or negative

slope the steepness of a line; the ratio of the vertical “rise” to the horizontal “run” between any two points on a line

slope-intercept formula an equation of a line of the form $y = mx + b$, where m is the *slope* and b is the *y-intercept*

square root if $p^2 = q$, then p is a square root of q ; if p is positive it is the principal square root of q , but if p is negative it is the minor square root of q

subset a subset of a set is a set whose elements are all contained in the set

sum the result of adding numbers or expressions together

system of equations two or more equations

T

terms the parts that are added to form an expression

trinomial a *polynomial* with three terms

U

union (of sets) the union of two or more sets is the set of elements that are in at least one of them; union is denoted by \cup

V

variable a letter that is used in place of a number

W

whole numbers the set of numbers 0, 1, 2, 3, ...; the set of all whole numbers is denoted \mathbb{W}

X

x -intercept the x -coordinate of a point where a graph meets the x -axis

Y

y -intercept the y -coordinate of a point where a graph meets the y -axis

Z

zero product property if the product of two *factors* is zero, then at least one of the factors must itself be zero

Formula Sheet

Axioms of the Real Number System

For any real numbers a , b , and c , the following properties hold:

Property Name

Addition

Multiplication

Closure Property:

$a + b$ is a real number

$a \times b$ is a real number

Identity Property:

$a + 0 = a = 0 + a$

$a \times 1 = a = 1 \times a$

Inverse Property:

$a + (-a) = 0 = -a + a$

$a \times a^{-1} = 1 = a^{-1} \times a$

Commutative Property:

$a + b = b + a$

$a \times b = b \times a$

Associative Property:

$(a + b) + c = a + (b + c)$

$(ab)c = a(bc)$

Distributive Property of

Multiplication over Addition:

$a(b + c) = ab + ac$ and $(b + c)a = ba + ca$

Properties of Equality

If $a = b$, then $a + c = b + c$. If $a = b$, then $ac = bc$.

If $a = b$, then $a - c = b - c$. If $a = b$, then $\frac{a}{c} = \frac{b}{c}$.

Absolute Value

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Order of Operations

Perform operations in the following order:

1. Anything in **grouping symbols** — working from the innermost grouping symbols to the outermost.
2. **Exponents**.
3. **Multiplications** and **divisions**, working from left to right.
4. **Additions** and **subtractions**, again from left to right.

Rules of Exponents

$$x^m \times x^n = x^{m+n}$$

$$x^m \div x^n = x^{m-n}$$

$$(x^m)^n = x^{mn}$$

$$x^1 = x$$

$$x^0 = 1$$

$$1^x = 1$$

$$\sqrt{x} = x^{\frac{1}{2}}$$

$$\sqrt[3]{x} = x^{\frac{1}{3}}$$

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$

$$x^{-m} = \frac{1}{x^m}$$

Using Roots

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Fractions

Adding and subtracting fractions with the same denominator:

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

Adding and subtracting fractions with different denominators:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

Multiplying fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Dividing fractions:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Applications Formulas

Investments

The return (I) earned in one year when p is invested at an interest rate of r (expressed as a fraction): $I = pr$

The total return (I) earned in one year when p_1 is invested at an interest rate of r_1 and p_2 is invested at an interest rate of r_2 :

$$I = p_1r_1 + p_2r_2$$

Mixtures

$$\text{concentration} = \frac{\text{amount of substance}}{\text{total volume}}$$

$$\text{percent of an ingredient} = \frac{\text{amount of ingredient}}{\text{total amount}} \times 100$$

The concentration (c) and total volume (v) of a mixture are given by where c_1 and v_1 describe the first ingredient and c_2 and v_2 the second.

$$cv = c_1v_1 + c_2v_2$$

Speed and Work Rate

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

$$\text{work rate} = \frac{\text{work completed}}{\text{time taken}}$$

Graphs

Slope of a line:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point-slope form:

$$y - y_1 = m(x - x_1)$$

Slope-intercept form:

$$y = mx + b$$

Standard form of a linear equation:

$$Ax + By = C$$

Two lines with slopes m_1 and m_2 are:

- parallel if $m_1 = m_2$
- perpendicular if $m_1 \times m_2 = -1$

Inequalities

$|x| < m$ means $-m < x < m$

$|x - c| < m$ means $c - m < x < c + m$

Properties of Inequality

For any real numbers a , b , and c :

If $a < b$, then $a + c < b + c$.

If $a < b$, then $a - c < b - c$.

For any real numbers a , b , and c :

If $a < b$ and $c > 0$, then $ac < bc$.

If $a < b$ and $c > 0$, then $\frac{a}{c} < \frac{b}{c}$.

Special Products of Binomials

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

Quadratics

Basic form of a quadratic equation:

$$ax^2 + bx + c = 0$$

The quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Completing the square:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$
$$ax^2 + bx + \frac{1}{a}\left(\frac{b}{2}\right)^2 = a\left(x + \frac{b}{2a}\right)^2$$

The discriminant:

$$b^2 - 4ac > 0 \Rightarrow 2 \text{ distinct real roots}$$

$$b^2 - 4ac = 0 \Rightarrow 1 \text{ real double root}$$

$$b^2 - 4ac < 0 \Rightarrow 0 \text{ real roots}$$

The vertex of $y = ax^2 + bx + c$:

$$\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$$

If $(x - a)(x - b) = 0$, then $x = a$ or $x = b$.

Index

A

abscissa 164
absolute values 19, 21, 128,
equations 128-132
inequalities 158-160
addends 14, 21
addition 14, 15, 21, 27, 28
of fractions 52, 53
of polynomials 266, 267, 272
property of equality 76, 77
property of inequalities 147, 148
of rational expressions 403-409
additive
identity 16
inverse 16, 266
age-related tasks 99, 251
algebraic expressions 9, 10
annual interest 108, 109
applications 89, 90
of fractional equations 417, 418
of inequalities 150-152
of polynomial division 294, 295
of quadratic equations 359, 360,
379-385
of systems of equations 249-251,
253, 255, 256, 258, 259
associative properties of addition and
multiplication 30, 34, 35
axioms 34, 35

B

bases 37, 38
binary operations 14, 15, 30
binomials 263
factors 310, 311
multiplication of 276, 281
products of 297-300
squaring of 342
braces 2
as grouping symbols 27
brackets as grouping symbols 27

C

canceling
common factors 46, 391, 392
fractions 284, 288, 289
closure of sets
under addition 14, 15, 21, 22, 34
under multiplication 14, 22, 34
coefficients 12
fractional coefficients 81, 82, 84, 85
coin tasks 92, 93

combining like terms 67, 68
common denominator 403, 404
common difference 95, 96
common factors 46, 47, 50, 51, 288, 289,
302, 303, 305-308, 313, 314, 322, 323,
329, 330, 391, 392
commutative properties of addition and
multiplication 24, 25, 30, 34, 35
completing the square 342-345, 347-353,
368-370
to derive the quadratic formula 355,
356
compound inequalities 155, 156, 158
conjunctions 155, 158
consecutive integer tasks 95, 96, 97,
150, 151
coordinate plane 164, 165
quadrants of 167-169
coordinates 164, 165, 179-181
coordinate of a point on a number line
18

D

decimal coefficients 86, 87
degree of a polynomial 264, 278
denominator 46, 288, 388
common denominator 403, 406-409
dependent systems of linear equations
239, 240
difference 24
difference of two squares 298, 325,
326
discriminant 372-378
disjunctions 156, 159, 160
distance 103-106
distributive laws 31
distributive property 34, 35, 67, 68, 70,
71, 276, 281
use in combining like terms 67
use in getting rid of grouping symbols
70, 84
dividend 291
division 24-26
of fractions 50, 51
of polynomials by monomials 283, 284
of polynomials by polynomials
285-292, 294, 295
property of equality 76-79
property of inequalities 139, 140,
142-145
property of square roots 43, 44
by rational expressions 397-401
by zero 17, 26, 33, 388

divisor 291, 303
domain of a relation 420-434
double root 375

E

economic tasks 383-385
elements of sets 2
elimination method for solving systems of
equations 244-248, 249-251, 255,
256, 258, 259
empty set 3
endpoints of intervals 137, 138, 155,
156, 158-160
equality
in equations 74
of functions 431-433
properties of
reflexive 14
symmetric 14
transitive 14
of sets 3
equations 74-76, 171, 194
absolute value equations 128-133
equivalent equations 76-79, 86
fractional equations 412-415, 417, 418
linear equations 74-78, 81, 82, 84-87,
89, 90, 175, 176, 229, 230, 249-250
systems of 229, 230
proof of an equation 74
quadratic equations 333-336, 338-341
setting up equations 95-97, 99-101,
103-106
solving equations 76, 77
of straight lines 170, 171, 173-175,
194, 195 205-207
equivalent
equations 76-79, 86, 87
fractions / rational expressions
46-48, 390-392, 406-410
inequalities 139, 140, 142-145
evaluating algebraic expressions 13
exponents 28, 37, 38, 73, 274, 275
fractional 38, 40
negative 286, 287
rules of 37, 38, 274, 275, 283-287
expressions
algebraic 13
numeric 9
quadratic 310, 311, 313-317, 318-320,
325-328

F

- factoring 288, 289 291, 292, 294, 295, 318-320, 322, 323, 333-336
 - monomials 302-303
 - polynomials 305-308, 310, 311, 313-317, 318-320, 322, 323, 325-330
 - polynomials using long division 291, 292
 - prime factorization 53, 54, 81
 - quadratic expressions and equations 294, 295, 310-320, 325-328, 333-336, 338-341
 - third-degree polynomials 322, 323
- factors 14, 22, 302, 303
 - common factors 288, 289, 302, 303, 305-308, 313, 314, 322, 323, 329, 330, 390-392
- fractional coefficients 81, 82, 84, 85
- fractional equations 412-415, 417, 418
- fractional powers 38, 40
- fractions (see also *rational expressions*)
 - 46, 47, 49, 50-54, 81, 82, 84, 85
 - equivalent 46, 47
- functions 424-427, 429-433
 - equality of 431, 432
 - function notation 429, 430

G

- gradient (slope) 189-192, 194, 195, 197-199, 201-203, 205-207, 209-210
- graph of a number 18
- graphing method for solving systems of equations 229, 230, 234
- graphing
 - inequalities on a coordinate plane 212-215, 217-220
 - inequalities on a number line 137-140, 155, 156, 170, 171, 173, 174
 - quadratic functions 362, 363, 365, 366, 368, 370
 - relations 420
 - straight lines 158-160, 175, 176, 186, 187
 - in the form $Ax + By = C$ 177, 178, 186, 187
 - systems of equations 229, 230, 237-240
 - systems of linear inequalities 222, 223
- gravity 379-381
- greatest common factor (GCF) 46, 47, 305-308, 391
- grouping like terms 67, 68
- grouping symbols 27, 70

H

- half-open intervals 138
- horizontal lines 173, 174

I

- identity 16
 - of addition 16
 - of multiplication 17
 - properties 30, 31
- inconsistent systems of linear equations 237, 238
- inequalities 137, 138
 - absolute value 158, 159, 160
 - applications of 139, 140, 142, 143
 - compound inequalities 155, 156
 - equivalent inequalities 150-153
 - graphing on a coordinate plane 212-215, 217-220, 222-225
 - graphing on a number line 137-140, 155, 156, 158-160
 - multistep inequalities 147-148
 - properties of
 - addition 139, 140, 147, 148
 - division 143, 145, 147, 148
 - multiplication 142, 144
 - subtraction 140
 - regions defined by 212-215, 217-220
- input-output tables 422
- inputs of binary operations 15
- integer problems 153
 - consecutive integer tasks 95-97, 150, 151
- integers 5
- interest 108-111
- interpreting results 379-381
- intersection of sets 7
- intervals 137-140, 155-160, 158-160
 - endpoints of 137-140, 158-160
- inverse 16, 17
 - operations 77-79
 - properties 34
 - under addition 17, 266
 - under multiplication 17, 285
- investment tasks 108-111
- irrational numbers 6
- isolating a variable 76-79

L

- least common multiple (LCM) 53, 81, 82, 84, 85, 407-410, 412-415
- like terms 67-68, 263, 264, 266, 267, 278
 - grouping like terms 68
- limitations on solutions 412
- linear equations 74-78, 81, 82, 84-87, 89, 90, 175, 176, 229, 230, 249-250
 - systems of 229, 230

lines

- curves (parabolas) 362, 364
- of symmetry 362, 368-370
- straight lines 170, 171, 173-184, 186, 187, 189-192, 194, 195, 197-199, 201-203
- long division 291-292, 294, 295
- lowest terms 46, 47

M

- mapping diagrams 421
- mathematical proofs 34, 35, 55-57
- minor square root 41
- mixture tasks 108-111, 113, 114, 116-119, 255, 256
- money tasks 92, 93
- monomials 263
 - dividing polynomials by 283, 284
 - factoring 302, 303
- motion tasks 103-106, 379-381
- multiplication 14, 15, 28, 29
 - of binomials 297-300
 - of fractions 49-51
 - of parentheses 148, 267, 268, 276-279, 318
 - of polynomials 267, 268, 276-281
 - property of -1 71
 - property of equality 78, 79
 - property of inequalities 143, 144
 - property of square roots 43, 44
 - by rational expressions 394-396
 - by zero 32, 33
- multiplicative
 - identity 16, 17
 - inverse 17
- multistep inequalities 147, 148

N

- natural numbers 5
- negative
 - exponents 286, 287
 - of a number 32, 33
 - numbers 18, 19
 - as real numbers 5
 - multiplying and dividing inequalities by negative numbers 144
 - terms 71
- nested grouping symbols 28
- nonlinear equations 175
- null set 3
- number line 18-20
 - to show inequalities 137-140, 155, 156, 158-160
- number system
 - rules of 14-22, 24-35
- numerator 46, 47, 288, 289, 388
- numeric expressions 9

O

one
 as the multiplicative identity 16, 17
 open intervals 137
 opposite
 of a number 16
 of a polynomial 266
 order of operations 27-29, 399, 400
 ordered pairs 164, 175, 179, 420, 421, 424
 ordinate 164
 origin 164

P

parabolas 362, 363
 parallel lines 197-199, 209
 parentheses 148
 as grouping symbols 27
 percent mixture problems 108-111, 113, 114, 116-119, 255, 256
 perfect square trinomials 327, 328, 342-345, 251-253, 353
 perpendicular lines 201-203, 210
 physical problems 379-381, 383-385
 plane 164, 165, 167-169
 quadrants of the coordinate plane 167
 plotting points 170, 177, 186
 point-slope form of equation of a line 189-192, 194, 195, 198, 203
 points of intersection 229
 points on a line 170, 175, 176, 179, 180
 points on a number line 18
 polynomials 263
 adding 266-268
 degree of 264, 278
 dividing polynomials by monomials 283, 284
 dividing polynomials by polynomials 288, 289, 291, 292
 factoring 305-308, 310, 311, 313-320, 322, 323, 325-330, 333-336
 factoring using long division 291, 292
 multiplying 276-281
 opposite of 266
 roots of 372-378
 simplifying 264
 subtracting 269-271
 positive numbers 18
 as real numbers 5
 postulates 34
 powers (exponents) 28, 37, 38, 73, 274, 275, 283, 284
 fractional 38, 40
 negative 286, 287
 rules of 37, 38, 274, 275, 283-287
 prime factorization 53, 81

prime factors 81, 302
 principal square root 40-42
 products 22
 of polynomials 276-279
 of powers 38
 special products of binomials 297-300
 proof of an equation 55-57, 74, 75
 properties of equality 14, 55-57, 76-79
 properties of inequalities 139, 140, 142-145

Q

quadrants of the coordinate plane 167
 quadratic equations 333
 graphing 362, 363, 365, 366, 368-370, 381
 quadratic expressions 310, 311
 factoring 310, 311, 313-316, 325-330
 in two variables 318-320
 solving by
 completing the square 338-341
 factoring 335, 336
 taking square roots 351-353
 using the quadratic formula 355-357, 359, 360, 368-370, 372-378
 quotients 25, 26
 of powers 38

R

radicals 40
 radicands 40
 range of a relation 420-422, 424-427, 429-433
 rates 121, 122, 124-126
 rate problems 103-106, 258, 259
 rational expressions
 adding and subtracting 403-410
 dividing by 397-401
 equations with 412-415, 417, 418
 equivalent 390-392, 406-408
 multiplying by 394-396, 397-401
 undefined 388, 389
 rational numbers 6
 real-life tasks
 using inequalities 150-153
 using quadratic equations 379-381, 383-385
 using systems of equations 388-392
 real numbers 5, 14-22, 24-35, 433
 reciprocals 17, 25, 26, 32, 33, 201, 202, 210, 397-401
 of polynomials 285-287
 reflexive property of equality 14

regions defined by inequalities 212-214, 217-220
 relations 420-422, 424, 425
 remainders after division 43, 288, 289, 291, 292
 removing fractions from equations 81, 82, 84, 85
 removing decimals from equations 86, 87
 return on investment 108
 "rise over run" formula 189-192, 197, 198
 roots 372-378
 of quadratic equations 76, 362, 363, 365, 366
 square roots 40-44, 338-341
 rules of exponents 37, 38, 73, 274, 275, 283, 284, 286
 rules of the number system 14-22, 24-35
 run 189

S

second-degree polynomials 318-320
 sequences of integers 95-97
 sets 2-6, 420-422, 424-426, 430-433
 sign of a number 19
 signs of coordinates in different quadrants 167
 simplifying
 algebraic expressions 13, 67, 68, 70, 71, 73-75, 147, 148, 150-153, 263, 264, 266-272, 274, 275, 281, 283-287
 inequalities 148
 numeric expressions 13
 rational expressions 390-392
 slope 189-192, 194, 197-199, 201-203, 205-207, 209, 210
 slope-intercept form of equation of a line 205-207, 209, 210
 solution intervals 137-140, 158-160
 solution sets 155, 160, 174, 212-215
 solving
 absolute value inequalities 158-160
 compound inequalities 155, 156, 158
 conjunctions 155, 156
 equations 76
 fractional equations 412-415
 inequalities 139, 140, 142-145, 147, 148, 150-153
 quadratic equations 333-336, 338-341, 351-353, 355-357
 speed 103-106
 square roots 40-42
 properties of 43, 44
 square root method for solving quadratics 338-341
 stacking method for multiplying polynomials 278, 279

straight lines 170, 171, 173-184, 186, 187, 189-192, 194, 195, 197-199, 201-203, 205-207

subsets 3

- subsets of the real numbers 5, 6

substitution method for solving systems of equations 232, 233, 235, 238, 240, 242, 249, 250, 253

substitution principle (or property) 15

subtraction 24, 25

- of fractions 52,
- of polynomials 269-272
- property of equality 76-78
- property of inequalities 140, 147-148
- of rational expressions 403-409

sum (of two numbers) 21

symmetric property of equality 14

systems of equations 229, 230, 244-248, 249-251, 253, 255, 256, 258, 259

- applications of 249-251, 253, 255, 256, 258, 259
- dependent systems 239, 240
- inconsistent systems 237, 238
- solving by elimination 244-248
- solving by graphing 229, 230, 237, 239, 240
- solving by substitution 232, 233, 238, 240

T

terms in algebraic expressions 67, 68

third-degree polynomials 322, 323

time-related problems 103-106

transitive property of equality 14

trial-and-error method for factoring quadratics 314-317

trinomials 263, 327, 328, 342-344

U

undefined rational expressions 388, 389

union of sets 7, 8

universal set 2

unknown quantities 9, 76

V

values

- of coefficients 12, 13
- of expressions 9, 10

variables 10, 67, 75, 76, 175

verifying points on a line 175, 176

vertex of a parabola 362, 363, 365, 366, 368-370

vertical line test for functions 426, 427

vertical lines 173, 198

W

whole numbers 5

word problems 89, 90, 92, 93, 95-97, 99-101, 103-106, 108-111, 113, 114, 116-119, 121, 122, 124-126, 150-153, 249-251, 253, 255, 256, 258, 259

work-related tasks 121, 122, 124-126

- work rates 122, 124-126

X

x -axis 164-169

x -intercepts 182, 183, 362, 363, 365, 366, 372-378

Y

y -axis 164, 165

y -intercepts 183, 184, 205-207, 362, 363, 365, 366, 370

Z

zero 16, 32, 33

- as the additive identity 16
- division by 17, 26, 32, 33, 388
- as a real number 5

zero product 33, 334-336